Lectures on Gauss-Bonnet

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May 30, 2005

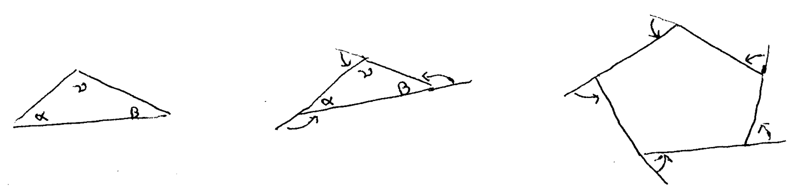
1 Statement of the Theorem in the Plane

According to Euclid, the sum of the angles of a triangle in the Euclidean plane is π. Equivalently, the sum of the *exterior angles* of a triangle is 2π, since the following formulas are the same:

$$(\pi - \alpha) + (\pi - \beta) + (\pi - \gamma) = 2\pi$$

$$\alpha + \beta + \gamma = \pi.$$

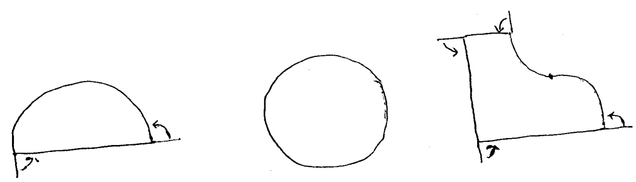
The advantage of the second formulation is that it works for figures with more sides: the sum of the exterior angles of any such figure is 2π.



The Gauss-Bonnet theorem is a generalization of this result to surfaces. As an initial stab at a generalization, we remain in the plane but allow the sides to curve. We will always traverse the boundary counterclockwise; that is how we know which exterior angle to draw. If T is the unit tangent to the sides, we choose N to be the unit normal obtained by rotating T counterclockwise by ninety degrees. Each edge along the boundary has a curvature κ, which could be positive or negative since we have fixed N. From now on, we denote the exterior angle at a vertex by $\triangle\theta$. Our preliminary Gauss-Bonnet theorem then asserts that

$$\sum_{\text{vertices}} \triangle\theta + \sum_{\text{edges}} \int \kappa = 2\pi$$

The three examples below illustrate this generalization. For the semicircle of radius R, the exterior angles are both $\frac{\pi}{2}$, the curved top has curvature $\frac{1}{R}$, and consequently $\int \kappa = \frac{1}{R}(\pi R) = \pi$. Notice that $\frac{\pi}{2} + \frac{\pi}{2} + \pi = 2\pi$.



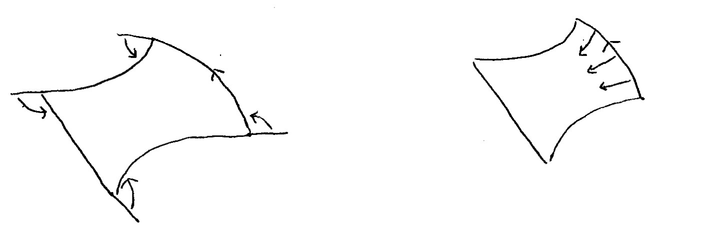
For the circle of radius R, the curvature is $\frac{1}{R}$ and consequently $\int \kappa = \frac{1}{R}$(length of circle). Since there are no exterior angles, our theorem asserts that $\frac{1}{R}$(length of circle) $= 2\pi$ and thus that

$$\text{the length of a circle of radius } R = 2\pi R.$$

Finally, in the third example, the contributions from the two semicircles on the right cancel out because one curvature is positive and one is negative, and the remaining exterior angles sum to 2π.

2 Statement of the Theorem on a Surface

We now generalize this theorem to a region in a coordinate patch on a surface. Let $\mathcal{R}$ be such a region, bounded by a finite number of paths meeting at vertices as in the picture below. Traverse this boundary counterclockwise. At each vertex, let $\triangle\theta$ be the exterior angle at that vertex. Parameterize each edge by arc length, and let N be the normal obtained by rotating the unit tangent T to the each counterclockwise by ninety degrees.



We are going to replace the curvature κ along the sides by a quantity called the *geodesic curvature*. It is the curvature of the edge as seen by a two-dimensional person. To define this, we use ideas from exercise set eight. Suppose $\gamma(t)$ is one of the edges, parameterized by arc length. Then $\frac{d}{dt} < \gamma', \gamma' >= \frac{d}{dt} 1 = 0$. By an application of the chain rule similar to that used in exercise set eight, $\frac{d}{dt} < \gamma', \gamma' >= \gamma' < \gamma', \gamma' >$ and this expression then equals $< \nabla_{\gamma'}\gamma', \gamma' > + < \gamma', \nabla_{\gamma'}\gamma' >$. So it follows that $\nabla_{\gamma'}\gamma'$ is perpendicular to $\gamma' = T$ and thus a multiple of the normal vector N. We denote this multiple by κ_g and call it the *geodesic curvature.*

Recall that the kth coordinate of $\nabla_{\gamma'}\gamma'$ is

$$\frac{d^2\gamma_k}{dt^2} + \sum_{ij} \Gamma^k_{ij} \frac{d\gamma_i}{dt}\frac{d\gamma_j}{dt}$$

This expression is zero for geodesics. Thus κ_g measures the edge's attempt to curve away from a geodesic and is zero for geodesics.

I'd recommend that readers not to worry about κ_g since all of our examples will involve regions whose boundary edges are geodesics with $\kappa_g = 0$.

Theorem 1 (Gauss-Bonnet) *If $\mathcal{R}$ is a region in the coordinate plane bounded by a finite number of edges traversed counterclockwise, and if κ_g is the geodesic curvatures of the sides and κ is the Gaussian curvature of the surface, then*

$$\sum_{vertices} \triangle\theta + \sum_{edges} \int \kappa_g + \int\int_{\mathcal{R}} \kappa = 2\pi$$

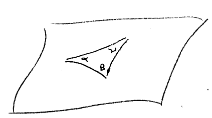
3 Examples

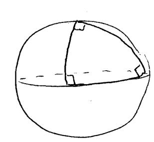
Theorem 2 *If a triangle has interior angles α, β, γ and geodesic sides,*

$$\alpha + \beta + \gamma - \pi = \int\int_{\mathcal{R}} \kappa$$

Proof: Alll $\kappa_g = 0$. So the statement follows from the Gauss-Bonnet theorem

$$(\pi - \alpha) + (\pi - \beta) + (\pi - \gamma) + \int\int_{\mathcal{R}} \kappa = 2\pi$$





This result explains how a two-dimensional person might determine the Gaussian curvature κ. Although κ is a function which varies from place to place, assume that κ is almost constant on a small triangle. Then $\int\int \kappa$ is approximately κ times the area of the triangle, and consequently

$$\kappa \sim \frac{\alpha + \beta + \gamma - \pi}{\text{area of triangle}}$$

Gauss actually tried this experiment, placing three observers on the tops of three mountains in Germany and accurately measuring the angles α, β, γ. The curvature of the earth is irrelevant here, because the sides were measured with light rays which presumably travel in a straight line. Gauss found that $\alpha + \beta + \gamma = \pi$ to within the accuracy of his measurements.

The right picture on the previous page shows a triangle covering one eighth of a sphere, with three right angles. If this sphere has radius R, its Gaussian curvature is $\frac{1}{R^2}$. The area of the triangle is one eighth of the area of a sphere, or $\frac{4\pi R^2}{8} = \frac{\pi R^2}{2}$. The Gauss-Bonnet theorem then asserts that $\frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} + \int\int \frac{1}{R^2} = 2\pi$ or $\frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} + \frac{1}{R^2}\frac{\pi R^2}{2} = 2\pi$, which is correct.

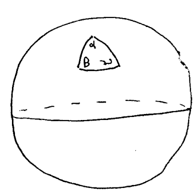
Theorem 3 *Suppose a triangle has geodesic sides and angles α, β, γ.*

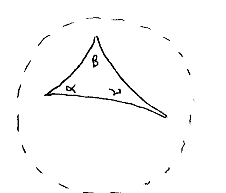
- *If the triangle is on a sphere of radius one,*

$$\textit{area of triangle} = \alpha + \beta + \gamma - \pi$$

- *If the triangle is in the Poincaire model of non-Euclidean geometry*

$$\textit{area of triangle} = \pi - (\alpha + \beta + \gamma)$$



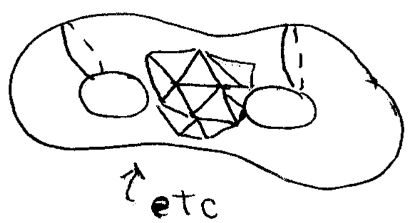


Remark: This follows from our previous result because $\kappa = 1$ on a sphere and $\kappa = -1$ on the Poincare disk. You proved the first statement directly in the first exercise of the course. As a corollary of the second statement, the area of any triangle in the Poincare disk is smaller than π.

4 The Global Version of the Theorem

If we combine the Gauss-Bonnet theorem with some easy topology, we obtain a spectacular global version of the theorem. Let our surface $\mathcal{S}$ be a doughnut with g holes. When $g = 0$ we have a sphere; when $g = 1$ we have an ordinary torus, and so forth.

Cut up the surface into vertices, edges, and faces; we will assume the faces are triangles. Let the numbers of these be V, E, F.



Theorem 4 (Euler-Poincare) *The number $V - E + F$ does not depend on the triangulation, but only on the topology of the surface $\mathcal{S}$. Indeed*

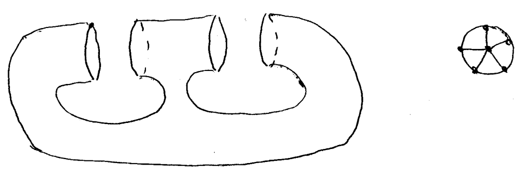
$$V - E + F = 2 - 2g$$

This theorem does not really require that the faces be triangles, but only that the faces be simply connected regions with no holes. The objects below look like a sphere to a topologist. Notice that $V - E + F = 8 - 12 + 6 = 2$ for the cube, and $4 - 6 + 4 = 2$ for the tetrahedron.

We can easily prove the theorem. We first prove it for the sphere, where the magic number $V - E + F$ is supposed to equal two. Triangulate a sphere and suppose its magic number is M. Rotate the sphere until the north pole is *inside* one of the triangles. Remove this triangle, so the magic number is now $M-1$. Then flatten the remaining part of the sphere into the plane. The result will be a cluster of triangles.

Begin removing these triangles one by one from the outside. You'll have to remove two types of triangles. If a triangle is like the shaded triangle on the left part of the cluster above, you'll remove one edge and one face, so the magic number will not change. If a triangle is like the shaded triangle on the right, you'll remove two edges, one vertex, and one face, and the magic number again does not change. In the end you'll have only one triangle, and the magic number will be $3-3+1=1$. So $M-1=1$ and $M=2$.

Now we prove the theorem for a general surface. Again triangulate this surface and call its magic number M. Cut the surface along each hole as indicated below; these cuts should be along edges of the triangulation. Notice that the magic number does not change, because along the circle cut we have a certain number n of vertices and the same number n of edges, and we just duplicated each such vertex and edge.



The result is a surface which is topologically a sphere, except that $2g$ disks have been removed from this sphere. Glue in triangulated disks to fill these missing disks. One such triangulated disk is shown on the right above. Notice that we have added an extra vertex, n extra edges, and n extra faces. So each such disk increases the magic number by one and in the end it equals $M+2g$. But now we have a sphere, and we know that its magic number is 2. So $M+2g=2$ and thus $M=2-2g$.

Now we combine this theorem with the Gauss-Bonnet theorem to obtain

Theorem 5 (Global Gauss-Bonnet Theorem) *If $\mathcal{S}$ is a surface with g holes, then*

$$\frac{1}{2\pi}\int\int_{\mathcal{S}}\kappa = 2-2g$$

Proof: Cut the surface into triangles. By the original Gauss-Bonnet theorem,

$$\int\int_{\text{triangle}} \kappa = (\alpha + \beta + \gamma) - \pi - \sum_{\text{edges}} \int \kappa_g$$

Sum this expression over all triangles to get

$$\int\int_{\mathcal{S}} \kappa = \sum_{\text{triangles}} (\alpha + \beta + \gamma) - \pi \sum_{\text{triangles}} 1 - \sum_{\text{triangles and edges}} \int \kappa_g$$

We can compute the first sum by summing the angles around a particular vertex and then summing over vertices. So the first sum is $2\pi V$. The second sum is πF. I claim that the third sum is zero. Indeed, each edge meets two triangles, and on both triangles the normal N has been chosen to point into the triangle. So the numbers κ_g differ by a sign from one triangle to the next. Thus the integrals have opposite signs and cancel, and we get

$$\int\int_{\mathcal{S}} \kappa = 2\pi V - \pi F$$

Notice that on a surface each triangular face has three edges. This gives $3F$ edges altogether, but since each edge is an edge of two triangles, we have counted each edge twice. So there are $\frac{3F}{2}$ edges and thus $-\frac{F}{2} = -\frac{3F}{2} + F = -E + F$. Therefore

$$\int\int \kappa = 2\pi \left(V - \frac{F}{2} \right) = 2\pi(V - E + F) = 2\pi(2 - 2g)$$

as desired.

5 Consequences of the Global Theorem

At this late stage, it is useful to recall the central philosophy of the course. We examined the geometry of surfaces from two points of view: from the viewpoint of a three-dimensional person looking at the surface from the outside, and from the point of view of a two dimensional person living on the surface. By now you probably know where my sympathies lie. I prefer the viewpoint of the two dimensional person, because I am such a person in one higher dimension.

Our two dimensional person starts with a coordinate system. Because the geometry on the surface need not be Euclidean, we cannot require that coordinates be Cartesian. (Actually, Descartes allowed skew coordinates; it was only later that the advantages of orthogonal coordinates became clear.) So instead we allow arbitrary coordinates. In Italy all roads lead to Rome, so I suppose the Italians like polar coordinates. In Boston the roads were originally cow paths and the city is laid out in cow-coordinates. Kansans believe in orthogonal coordinates and feel right at home in San Francisco where the streets were apparently drawn with a ruler by a fellow who never went outside and noticed the hills.

We need additional information to describe distances and angles. That additional information is encoded in the g_{ij} metric tensor. This tensor depends on the coordinate system. When we vary the g_{ij}, we get different geometries on our surface, although many variations are just coordinate changes for the same geometry. Riemann's counting argument shows that there is one function, κ, which tells us which geometry we have inherited.

Where do the g_{ij} come from? There are several possible answers. One answer is that our surface is *really* curved into the third dimension, and the g_{ij} just come from Euclidean geometry in that third dimension. That was our original point of view in this course. But our surface might also be curved into more than three dimensions. This gives additional geometries on the surface. For instance, all geometries on a torus inside R^3 have points where $\kappa > 0$ and other points where $\kappa < 0$. But we can also embed the torus into R^4 in such a manner that $\kappa = 0$ everywhere.

Another possibility is that the g_{ij} don't come from any embedding, but rather from physics. In this point of view, the g_{ij} are like the electromagnetic field; they are created by moving masses and the behavior of these masses on other particles is best described by saying that the other particles travel along geodesics. If this is your point of view, you might believe that the masses are creating a *real geometric change* g_{ij}, or you might believe that the geometry is always Euclidean and it is just convenient to describe the forces in terms of a fictitious g_{ij}. It doesn't really make sense to argue about which point of view is correct because the predictions are the same in either case.

Finally, I suppose you might imagine that the g_{ij} are created artificially by a society, perhaps by law. Imagine our surface $\mathcal{S}$ is the universe described in Star Wars, and the governing council has decided to organize life by imposing a geometry on this universe. To simplify commerce and education, the council would probably require that the geometry be the same everywhere. That way, planet X wouldn't have spherical trig books while planet Y has ordinary trig books. If the geometry is the same everywhere, then κ is a constant. So the governing council might decree "From now on, geometry everywhere is Euclidean. You are free to adopt any local coordinates you like, but your g_{ij} must have curvature zero." Instead of fighting over whether somebody's Father is evil, the council could fight over a real issue, like whether to discard all the old fashioned Euclidean books and adopt spherical geometry throughout the University. Now that would be a movie!

However, the Gauss-Bonnet theorem imposes a giant restriction on all of this fantasy. We are not after all free to determine the g_{ij} and thus κ arbitrarily. In all cases it must be true that $\int\int_{\mathcal{S}} \kappa = 2 - 2g$. If we live on a sphere, then $\int\int \kappa$ must be positive and thus there must be some places where $\kappa > 0$. If we live on a torus, then $\int\int \kappa$ must be zero, and therefore if there are places where $\kappa > 0$, there must also be places where $\kappa < 0$. If we live on a surface with at least two holes, then $\int\int \kappa$ must be negative, and therefore there must be points where $\kappa < 0$.

In our Star Wars fantasy (making the rather big assumption that the Universe is two dimensional and compact!), the council can only choose one geometry of constant curvature. It doesn't matter how much power they have; mathematics constrains them.

Let me bring this back down to earth and prove a couple of theorems.

Theorem 6 *A surface with g holes can be embedded in some high dimensional R^n to yield ordinary Euclidean geometry everywhere if and only if the surface is a doughnut with one hole.*

Proof: If the geometry is Euclidean, then $\kappa = 0$ and so $\int\int \kappa = 0 = 2 - 2g$, so $g = 1$. Conversely, consider a torus. The standard torus in R^3 has places where $\kappa > 0$ and other places where $\kappa < 0$. But we can embed a torus into R^4 so $g_{ij} = \delta_{ij}$ and thus $\kappa = 0$. Indeed, think of a torus as a square with sides of length 2π and opposite sides identified. Map the torus to R^4 by $(u, v) \to (\cos u, \sin u, \cos v, \sin v)$. Notice that opposite sides map to the same point, so this is well defined. Then $\frac{ds}{du} = (-\sin u, \cos u, 0, 0)$ and $\frac{ds}{dv} = (0, 0, -\sin v, \cos v)$ and so $g_{11} = 1, g_{12} = 0, g_{22} = 1$. Thus the inherited geometry is Euclidean.

Theorem 7 *If a surface with g holes is embedded in R^3, then there are points where $\kappa > 0$. Consequently the surface does not have constant curvature unless it is a sphere.*

Proof: Find a point p on the surface as far as possible from the origin. Draw a sphere centered at the origin through this point. Suppose this sphere has radius R. Then the principal curvatures of the sphere at p are both equal to $\frac{1}{R}$ and clearly the principal curvatures of the surface at p are even larger. So $\kappa > 0$ at p.

Theorem 8 *Every surface with g holes can be given a geometry with constant curvature, and thus a geometry with curvature one, zero, or minus one. We do not claim that this geometry arises from an embedding into some high dimensional R^n, although Nash proved that it does. Moreover*

- *If the surface is a sphere, then the curvature must be positive. If it is one, then the area of the surface must be 4π.*
- *If the surface is a torus, then the curvature must be zero.*
- *If the surface is has at least two holes, then the curvature must be negative. If it is minus one, then the area of the surface must be $4\pi(g - 1)$.*

Proof: If κ is constant, then

$$\frac{1}{2\pi}\int\int \kappa = \frac{1}{2\pi}\kappa \text{ (area of surface)} = 2 - 2g$$

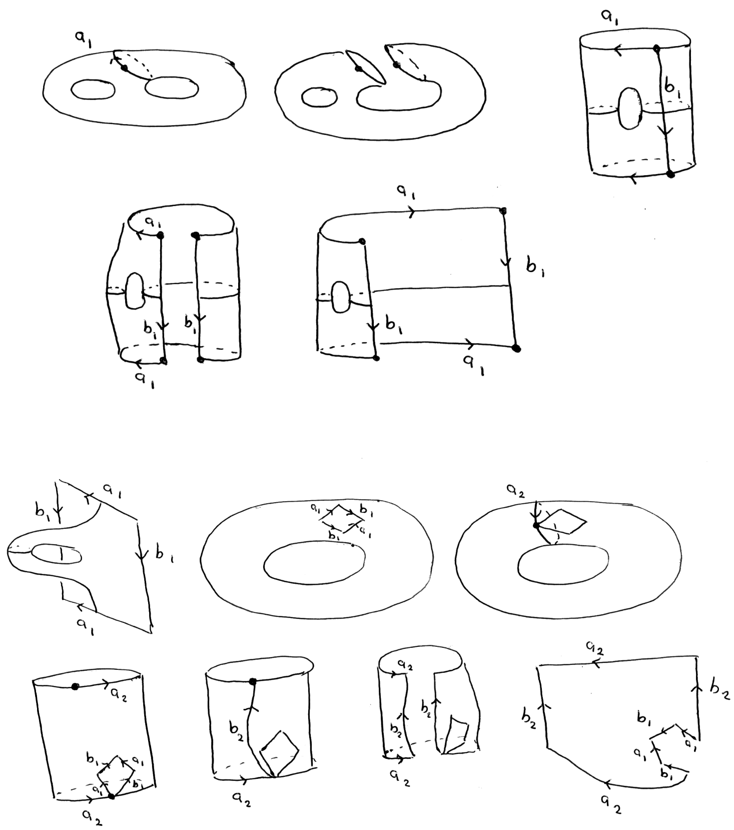
and so

$$\kappa(\text{area of surface}) = 4\pi(1-g)$$

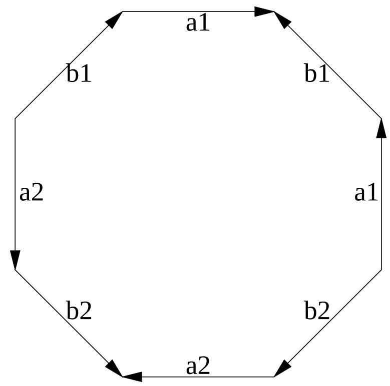
So $\kappa > 0$ implies $(1-g) > 0$ and so $g = 0$, and $\kappa = 0$ implies $g = 1$, and $\kappa < 0$ implies $(1-g) < 0$ and so $g \geq 2$.

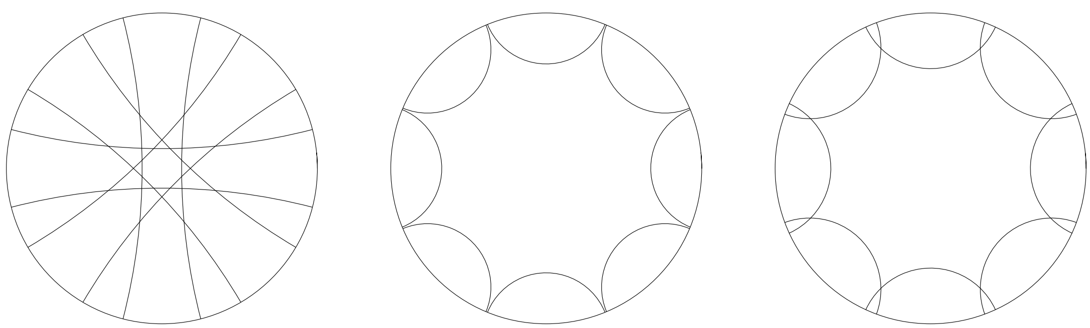
It remains to prove that each surface does have such a constant curvature metric. We have already done this for the sphere and torus. So consider a surface with $g \geq 2$.

Our argument depends on a result in topology. A surface with g holes can be constructed from a regular polygon with $4g$ sides by labeling the sides counterclockwise as $a_1, b_1, a_1^{-1}, b_1^{-1}, \ldots, a_g, b_g, a_g^{-1}, b_g^{-1}$ and then gluing the a-sides and b-sides together with arrows matching. When this is done, all vertices of the polygon glue together. The picture below shows this for the case $g = 2$, but requires a lot of imagination. See a topology book for better pictures.



To finish the argument, we draw such a polygon in the Poincare model of non-Euclidean geometry. Notice that the sides are geodesics in this geometry. We can glue the sides together preserving distance, and then the g_{ij} match correctly across edges, giving coordinates along the edges where half of the coordinate comes from one side of a_i and the other half from one side of a_i^{-1}.. To finish the argument, it suffices to show that there are good coordinates near the vertex obtained when we glue all corners of the polygon together.



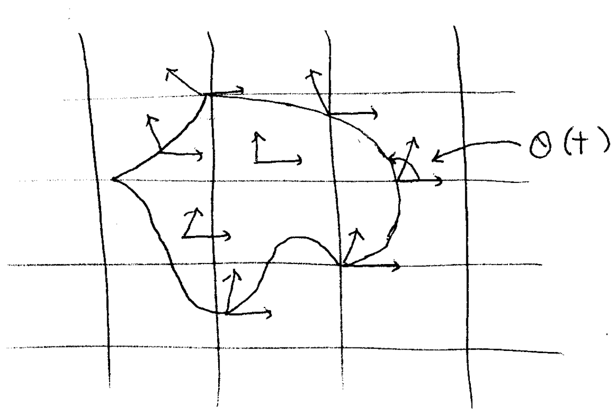


Since all vertices glue together, we must show that the total angle around this vertex is 2π. The picture on the left shows that when we make a very small octagon near the center where geometry is almost Enclidean, the angle sum is approximately the angle sum of a Euclidean octagon, which is $6\pi > 2\pi$. When we make a very large octagon, the angle sum is almost zero. So between these extremes there is a spot where the angle sum is exactly 2π.

6 Proof of Gauss-Bonnet Part 1

The proof of the Gauss-Bonnet theorem is very beautiful. It would be nice if everyone understood it, but I am aiming this last section at the graduate students because they are taking an analysis course whose central topic is differential forms and Stokes formula and proof of the Gauss-Bonnet theorem comes from a simple application of Stokes formula.

We begin the proof by using the Gram-Schmidt process to construct an orthonormal basis $\{e_1, e_2\}$ at each point of our coordinate system. It is *not true* that $e_1 = \frac{\partial}{\partial u}$ and $e_2 = \frac{\partial}{\partial v}$; instead e_1 and e_2 have been constructed from these basis vectors by twisting until they become orthonormal. The $\{e_i\}$ provide a *moving frame* over our coordinate system, much like the moving frame analyzed for curves by the Frenet-Serret formulas.



Once we have this basis, we can assign an angle $\theta(t)$ to each point on the boundary curves of our region, giving the angle that the tangent to the curve makes with e_1. Of course several different angles can be defined differing by multiples of 2π, but we can choose the angles uniquely by requiring that they vary continuously and increase by the exterior angle $\triangle\theta$ at vertices. See the picture above.

I now claim that

$$\sum_{\text{vertices}} \triangle\theta + \sum_{\text{edges}} \Big(\theta(\text{end}) - \theta(\text{beginning})\Big) = 2\pi$$

and so

$$\sum_{\text{vertices}} \triangle\theta + \sum_{\text{edges}} \int \frac{d\theta}{dt} = 2\pi$$

Indeed, when we traverse the boundary once counterclockwise and return to our starting point, the angle which γ' makes with e_1 at the start and end must be the same, and thus the change in θ over our travels must be a multiple of 2π. If we gradually deform our boundary curve, this total angle change must also deform gradually. But since it is always a multiple of 2π, it cannot change at all during the deformation. Deform the boundary curve until it is just a small circle. Then the total change in θ is clearly 2π, so it must always be 2π.

7 Proof of Gauss-Bonnet Part 2

Parameterize each boundary curve by arc length. Then $\gamma'(t)$ has length one. Since $\gamma'(t)$ makes angle $\theta(t)$ with e_1, we have

$$\gamma'(t) = \cos\theta(t) e_1 + \sin\theta(t) e_2$$

Recall from page three that the normal N to these boundary curves is obtain by rotating T ninety degrees counterclockwise, so $N = -\sin\theta(t)e_1 + \cos\theta(t)e_2$. By definition of κ_g we have $\nabla_{\gamma'}\gamma' = \kappa_g N$. The derivative of γ' involves two terms, one obtained by differentiating $\theta(t)$ and leaving the e_i fixed, and one obtained by leaving θ fixed and differentiating the e_i. So

$$\begin{aligned}\nabla_{\gamma'}\gamma' &= \frac{d\theta}{dt}\Big(-\sin\theta \; e_1 + \cos\theta \; e_2\Big) + \Big(\cos\theta \; \nabla_{\gamma'} e_1 + \sin\theta \; \nabla_{\gamma'} e_2\Big) \\ &= \kappa_g N = \kappa_g\Big(-\sin\theta(t) e_1 + \cos\theta(t) e_2\Big)\end{aligned}$$

Since every expression in sight except one is a multiple of $-\sin\theta(t)e_1 + \cos\theta(t)e_2$, that expression must also be such a multiple. Dotting everything with N gives

$$\frac{d\theta}{dt} + \Big(< \cos\theta \; \nabla_{\gamma'} \; e_1 + \sin\theta \; \nabla_{\gamma'} \; e_2, -\sin\theta \; e_1 + \cos\theta \; e_2 > \Big) = \kappa_g$$

Insert this formula into the last formula on page twelve, to obtain

$$\sum_{\text{vertices}} \triangle\theta + \sum_{\text{edges}} \int \kappa_g - \sum_{\text{edges}} \int \Big(< \cos\theta \; \nabla_{\gamma'} e_1 + \sin\theta \; \nabla_{\gamma'} e_2, -\sin\theta(t) e_1 + \cos\theta(t) e_2 > \Big) \; = \; 2\pi$$

This result is very close to the Gauss-Bonnet theorem. We must replace the last integral over the edges bounding our region by

$$\int\int_{\mathcal{R}} \kappa$$

This will be done using Stokes' formula, or rather the special case usually called Green's formula.

8 Proof of Gauss-Bonnet Part 3

In the differential geometry of curves, we obtained the Frenet-Serret formulas by measuring the change in the moving frame as we moved along the curve. We are going to do the same thing with our moving frame $\{e_1, e_2\}$. But this time we can move in several different directions. So by analogy with the equations

$$\frac{d}{ds} X_i = \sum_j a_{ij} X_j$$

on page eight of the lecture results, let us write

$$\nabla_X e_i = \sum_j \omega_{ij}(X) e_j$$

Our expression contains a vector X because we can differentiate in many different directions. Our sums only go from $j = 1$ to $j = 2$, so this equation is a fancy way of writing

$$\nabla_X e_1 = \omega_{11}(X) e_1 + \omega_{12}(X) e_2$$
$$\nabla_X e_2 = \omega_{21}(X) e_1 + \omega_{22}(X) e_2$$

According to the Frenet-Serret formulas, $A^T = -A$. For exactly the same reason, we now have $\omega(X)^T = -\omega(X)$. Said another way, $\omega_{11}(X) = \omega_{22}(X) = 0$ and $\omega_{21}(X) = -\omega_{12}(X)$. The proof is easy: $< e_i, e_j >= \delta_{ij}$ and so $X < e_i, e_j >= 0$, so $< \nabla_X e_i, e_j > + < e_i, \nabla_X e_j >= 0$. Therefore

$$< \sum \omega_{ik}(X)\ e_k, e_j >=< e_i, \sum \omega_{jk}(X)\ e_k >= 0$$

and thus

$$\omega_{ij}(X) + \omega_{ji}(X) = 0$$

Turn back to the formula on page thirteen. The complicated term at the end is

$$- \sum_{\text{edges}} \int < \cos\theta\ \nabla_{\gamma'}\ e_1 + \sin\theta\ \nabla_{\gamma'}\ e_2, -\sin\theta\ e_1 + \cos\theta\ e_2 >$$

$$= - \sum_{\text{edges}} \int < \cos\theta\ \omega_{12}(\gamma')\ e_2 + \sin\theta\ \omega_{12}(\gamma')\ e_1, -\sin\theta\ e_1 + \cos\theta\ e_2 >$$

Since the e_i are orthonormal, this expression reduces to the following expression, which we want to show is the same as $\int \int \kappa$:

$$= - \sum_{\text{edges}} \int < \omega_{12}(\gamma') e_1, e_2 >$$

9 Proof of Gauss-Bonnet Part 4

Recall Green's theorem. Suppose $E = \Big(E_u(u,v), E_v(u,v)\Big)$ is a vector field in the plane. Let γ be a curve which surrounds a region $\mathcal{R}$ in the counterclockwise direction. We can form the line integral of the vector field E around this curve. According to Green:

$$\int_\gamma \Big(E_u, E_v\Big) \cdot d\gamma = \int\int_{\mathcal{R}} \left(\frac{\partial E_v}{\partial u} - \frac{\partial E_u}{\partial v}\right) du dv$$

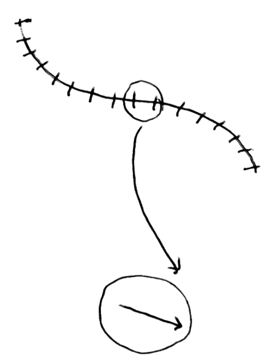
We are going to apply this theorem. But if we write everything in coordinates as the theorem suggests, we get a horrible mess involving the g_{ij} and their derivatives, the Christoffel symbols, and the kitchen sink. The trick is to understand this theorem from a more conceptual point of view. Then the calculation we must perform becomes very easy.

In this section we start with the left side, and thus with line integrals. Why do we compute line integrals of vector fields, rather than some more straightforward integral?

Suppose we want to integrate an object ω over curves, but we do not yet know what kind of object ω should be. To integrate, we divide our curve into small pieces, compute ω on each piece, add, and take a limit. So ω should be an object which gives a number when evaluated on a small piece of the curve. If $\gamma(t)$ is our curve, the small piece from t to $t + \Delta t$ is approximately $\gamma'(t)\Delta t$. We expect the world to linearize when we make very small approximations, so

$$\omega\Big(\gamma'(t)\Delta t\Big) = \omega\Big(\gamma'(t)\Big)\Delta t.$$

We conclude that ω should be an object which maps vectors to real numbers.



We could then write $\omega(X)$ in matrix language as

$$\omega(X) = \begin{pmatrix} \omega_1 & \omega_2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

and the line integral would become

$$\int_\gamma \omega = \int \omega \left(\frac{d\gamma}{dt} \right) \, dt = \int_a^b \left(\omega_1 \frac{d\gamma_1}{dt} + \omega_2 \frac{d\gamma_2}{dt} \right) \, dt$$

Notice that we recover the line integral notation that we started with, replacing E_u and E_v by ω_1 and ω_2. But our interpretation is different: ω is no longer a vector field; instead it is a map which assigns numbers to vectors. Such a map is called a *one form* and I will call it that from now on.

Incidentally, a side comment for graduate students. Many authors write vectors using $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ as a basis, so $X = X_1 \frac{\partial}{\partial u} + X_2 \frac{\partial}{\partial v}$. They then define du and dv to be the dual basis of the dual space. In this language

$$\omega = \omega_1 \; du + \omega_2 \; dv$$

Let us return to the Gauss-Bonnet theorem. Notice that $\omega_{12}(X) =< \nabla_X e_1, e_2 >$ is indeed a linear map from vectors X to real numbers. Thus $\omega_{12}(X)$ is a one-form which we will just call ω. So the theorem on the bottom of page 13 can now be written in the suggestive language

$$\sum_{\text{vertices}} \triangle\theta + \sum_{\text{edges}} \int \kappa_g - \int_{\partial \mathcal{R}} \omega = 2\pi$$

This is looking better and better. But It would be a bad idea to write ω in coordinates. It has a nice definition in terms of e_1 and e_2, but e_1 and e_2 are not our coordinate vectors. If we write in terms of coordinates, we need formulas for e_1 and e_2 in terms of $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ and believe me, you don't want to see those formulas.

10 Proof of Gauss-Bonnet Part 5

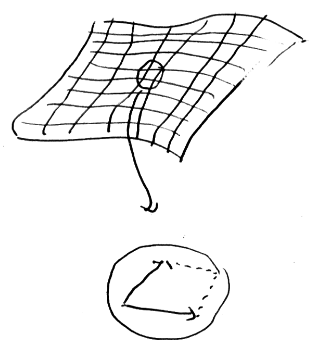
We need a similar conceptual point of view for the right side $\int\int \left(\frac{\partial E_v}{\partial u} - \frac{\partial E_u}{\partial v} \right) dudv$ of Green's theorem. For the moment we do not care that this integrand comes from E, so the right side is really just a function of two variables. Let's switch to modern notation and call this function $\Omega(u, v)$. We'll then write the integral on the right side of Green's theorem as $\int\int \Omega(u, v) \; dudv$. The graduate students may prefer the notation $\int\int \Omega(u, v) \; du \wedge dv$.

This integral is so straightforward that you may suspect I have nothing to add. But there is a great deal to add. At first this is going to seem abstract and pointless and you'll think I'm making a mountain out of a molehill.

I'm going to argue that Ω is not really a function; instead it is what is called a two-form. It is easier to explain this deeper point of view if we examine the three dimensional version of Green's theorem, which is called Stokes' theorem.

$$\int_{\partial\mathcal{S}} E \cdot \frac{d\gamma}{dt}\, dt = \int\int_{\mathcal{S}} \mathrm{curl} E \cdot n \; d\mathcal{S}.$$

The expression on the right is a *surface integral*. Surface integrals are not integrals of functions; instead they are integrals of vector fields. But I'll argue that they are *really* integrals of a new kind of object called a two-form. Suppose we want to integrate an object Ω over a surface, but we do not yet know what kind of object Ω should be. To integrate, we divide the surface into small pieces, compute Ω on each piece, add, and take a limit.



So Ω should be an object which gives a number when evaluated on a small piece of surface. Small parallelograms on the surface are defined by pairs of vectors X and Y. We expect the world to linearize when we make very small approximations, so Ω should be a real valued function defined on pairs of vectors: $\Omega(X, Y)$. This function should be linear in each variable separately if the other variable is held fixed.

Surface integrals depend on an orientation of the surface; changing the orientation changes the sign of the integral. So we expect that $\Omega(Y, X) = -\Omega(X, Y)$. *By definition,* a two-form is an assignment to each point of space of a map $\Omega(X, Y)$ which assigns to each pair of vectors a real number, and satisfies $\Omega(Y, X) = -\Omega(X, Y)$.

The surface integral of such a two-form on R^3 is given by the above picture, which easily translates into the precise formula

$$\int\int_{\mathcal{S}} \Omega = \int\int_{\mathcal{S}} \Omega\left(\frac{\partial s}{\partial u}, \frac{\partial s}{\partial v}\right)\ dudv$$

With this insight, we return to the double integral which appears on the right side of Green's theorem. In this special two-dimensional case, a bilinear and skew-symmetric map is completely determined by one coordinate function, which is why Ω looked like a function originally. Indeed

$$\Omega\left(X_1\frac{\partial}{\partial u} + X_2\frac{\partial}{\partial v}, Y_1\frac{\partial}{\partial u} + Y_2\frac{\partial}{\partial v}\right) = \ (X_1 Y_2 - X_2 Y_1)\,\Omega\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right)$$

So a two-form on the plane is completely determined by the expression $\Omega\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right)$, which is just a function of u and v. Moreover, the integral of a two-form in the plane is then clearly given by

$$\int\int_{\mathcal{S}} \Omega = \int\int_{\mathcal{S}} \Omega\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right)\ dudv$$

Since $\Omega\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right)$ is a function, this integral reduces to the original double integral of a function.

When we summarize the results of the previous two sections, it looks like we have accomplished nothing except to rename a few coefficients. Namely, Green's theorem now takes the form

$$\int_{\partial\mathcal{R}} (\omega_1\ du + \omega_2\ dv) = \int\int_{\mathcal{R}} \Omega\ dudv$$

where

$$\Omega\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) = \frac{\partial \omega_2}{\partial u} - \frac{\partial \omega_1}{\partial v}$$

11 Proof of Gauss-Bonnet Part 6; the d Operator

There is one more piece of information in Green's theorem. The theorem requires a map which takes the one-form ω to the two-form Ω. In modern mathematics this map is called the d-operator and we write $\Omega = d\omega$. In coordinates the map is given by the simple formula

$$\Omega(u,v) = d\omega(u,v) = \frac{\partial \omega_2}{\partial u} - \frac{\partial \omega_1}{\partial v}$$

Using this map, we can write Green's theorem as follows:

$$\int_{\partial\mathcal{R}} \omega = \int\int_{\mathcal{R}} d\omega$$

You probably suspect that I cannot add insight to such a simple formula. But I can. The trouble with the formula is that it is not coordinate invariant. Here is the deeper version of this formula:

Theorem 9 *Let X and Y be vector fields and let ω be a one-form. Then*

$$d\omega(X,Y) = X\Big(\omega(Y)\Big) - Y\Big(\omega(X)\Big) - \omega([X,Y])$$

Proof: Let us first verify this when $X = \frac{\partial}{\partial u}$ and $Y = \frac{\partial}{\partial v}$. Then

$$X\Big(\omega(Y)\Big) - Y\Big(\omega(X)\Big) - \omega([X,Y]) = \frac{\partial}{\partial u}\omega_2 - \frac{\partial}{\partial v}\omega_1 - \omega\left(\left[\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right]\right)$$

$$= \frac{\partial \omega_2}{\partial u} - \frac{\partial \omega_1}{\partial v} \text{ (since mixed partial derivatives commute)}$$

This is indeed the coordinate version of the formula for the d operator given earlier.

To finish the proof, it suffices to show that the expression $X(\omega(Y)) = Y(\omega(X)) - \omega([X,Y])$ is linear over functions, since $d\omega(X,Y)$ is certainly linear over functions. Indeed if both sides are linear over functions, we could pull the functions out and compare both sides on basis vectors, where we've already proved that they are equal.

Since $X(\omega(Y)) = Y(\omega(X)) - \omega([X,Y])$ is skew symmetric, it suffices to prove that when we multiply X by f, the expression just multiplies by f. Preliminarily notice that

$$[fX,Y](g) = f(X(Y(g)) - Y(fX(g)) = f(X(Y(g)) - Y(f)X(g) - fY(X(g))$$

and so

$$[fX,Y] = f[X,Y] - Y(f)X$$

Hence

$$fX(\omega(Y)) - Y(\omega(fX)) - \omega([fX,Y]) = fX(\omega(Y)) - Y(f\omega(X)) - \omega(f[X,Y] - Y(f)X)$$

$$= fX(\omega(Y)) - Y(f)\omega(X) - fY(\omega(X)) - f\omega([X,Y]) + Y(f)\omega(X)$$

$$= f\Big(X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])\Big)$$

QED.

12 Proof of Gauss-Bonnet Part 7

All of this preparation makes the following calculation easy:

Theorem 10 *Let $\omega(X)$ be the one form*

$$\omega_{12}(X) = < \nabla_X e_1, e2 > .$$

Then $d\omega$ is the two form given by the following formula in terms of the Riemann curvature tensor:

$$d\omega(X,Y) = R(X,Y,e_1,e_2)$$

Proof: We have

$$d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]) = X < \nabla_Y e_1, e_2 > -Y < \nabla_X e_1, e_2 > - < \nabla_{[X,Y]} e_1, e_2 >$$

$$= < \nabla_X \nabla_Y e_1, e_2 > + < \nabla_Y e_1, \nabla_X e_2 > - < \nabla_Y \nabla_X e_1, e_2 > - < \nabla_X e_1, \nabla_Y e_2 > - < \nabla_{[X,Y]} e_1, e_2 >$$

But by section eight, $\nabla_Y e_1$ is a multiple of e_2 and $\nabla_X e_2$ is a multiple of e_1. So the second term is zero because e_2 and e_1 are perpendicular. The fourth term is similarly zero, and we are left with

$$< \nabla_X \nabla_Y e_1, e_2 > - < \nabla_Y \nabla_X e_1, e_2 > - < \nabla_{[X,Y]} e_1, e_2 >= R(X,Y,e_1,e_2)$$

Remark: It is now easy to complete the proof of the Gauss-Bonnet theorem. Recall that on page 16 we obtained

$$\sum_{\text{vertices}} \triangle\theta + \sum_{\text{edges}} \int \kappa_g - \int_{\partial\mathcal{R}} \omega = 2\pi$$

By Green's theorem, the last term is equal to

$$-\int\int_{\mathcal{R}} R\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}, e_1, e_2\right)$$

Let us write

$$\frac{\partial}{\partial u} = \alpha e_1 + \beta e_2$$
$$\frac{\partial}{\partial v} = \gamma e_1 + \delta e_2$$

Then this double integral becomes

$$-\int\int_{\mathcal{R}} R\Big(\alpha e_1 + \beta e_2, \gamma e_1 + \delta e_2, e_1, e_2\Big) = -\int\int_{(} \alpha\delta - \beta\gamma) R(e_1,e_2,e_1,e_2) = \int\int((\alpha\delta - \beta\gamma)\kappa$$

Taking fancy dot products gives

$$g_{11} =< \frac{\partial}{\partial u}, \frac{\partial}{\partial u} >= \alpha^2 + \beta^2$$

$$g_{12} =< \frac{\partial}{\partial u}, \frac{\partial}{\partial v} >= \alpha\gamma + \beta\delta$$

$$g_{22} =< \frac{\partial}{\partial v}, \frac{\partial}{\partial v} >= \gamma^2 + \delta^2$$

which can be written in matrix form as

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$$

It follows that

$$\det(g_{ij}) = (\alpha\delta - \beta\gamma)^2$$

and so

$$\int\int (\alpha\gamma - \beta\delta)\kappa = \int\int \sqrt{\det g_{ij}}\ \kappa$$

This last integral is by definition the integral of the *function* κ over $\mathcal{R}$, where the factor $\sqrt{\det g_{ij}}$ is present to compensate for the fact that area of small pieces of the surface is not *du dv* but rather has an extra factor due to the fancy inner product.

In the end, the formula on page 16 becomes

$$\sum_{\text{vertices}} \triangle\theta + \sum_{\text{edges}} \int \kappa_g + \int\int_{\mathcal{R}} \kappa\ \sqrt{\det g_{ij}}\ du dv = 2\pi$$

which is exactly the Gauss-Bonnet formula.

QED.