Review 1

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1 Curves

From the chapter on curves, you should know

1. the formula for arc length in section 1.2;
2. the definition of $T(s), \kappa(s), N(s), B(s)$ in section 1.4.
3. the fact that $\kappa = \frac{1}{R}$ where R is the radius of the osculating circle, on the middle of page 7;
4. The proof of the Frenet-Serret theorem, given in section 1.6 (know this section from the start on page 8 to the end of page 10);
5. The fundamental theorem of curve theory, and its proof. This is in sections 1.8 and 1.9 and can be done either using the existence and uniqueness theorem for systems of differential equations, or by computer.

Other sections of chapter one can be skipped. Please concentrate on the proof of the Frenet-Serret formulas, and their use to prove the fundamental theorem.

Exercise 1: The curve $\gamma(s) = \left(\cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right)$ is parameterized by arc length. Compute $T(s), N(s), B(s)$, $\kappa(u)$, and $\tau(s)$.

Exercise 2: Suppose $\gamma(u)$ is parameterized by arc length. Prove that $\frac{d\gamma}{du}$ has length one.

Exercise 3: Suppose $\gamma(u)$ is parameterized by arc length. Prove that $\frac{dT}{du}$ is perpendicular to T.

Exercise 4: Compute $\frac{d^2T}{du^2}$ and $\frac{d^2N}{du^2}$ in terms of T, N, and B.

Remark: In this section we discovered how to find the radius and center of the osculating circle for a curve. I want you to be able to do this on the midterm; it is the only deep calculation I'm likely to ask you to do.

Below are exercises which explain what to do. If the curve is parameterized by arc length, the calculation is straightforward, as in the first exercise. Otherwise you may need the hints at the end of this section.

Exercise 5: Find the center of the osculating circle for the helix given in exercise one at the point $s = \sqrt{2}\ \frac{\pi}{4}$.

Exercise 6: Consider the curve $\gamma(t) = (t, t^2, 0)$. Draw this curve. Find the center and radius of the osculating circle at $t = 0$.

Exercise 7: Consider the curve $\gamma(t) = (t, t^2, t^3)$. Draw this curve. Find the center and radius of the osculating circle at $t = 1$.

Hint: The last two exercises may be more difficult. Here is a hint. Call the original curve $\tau(t)$ and let $\gamma(s)$ be this curve parameterized by arc length. Then $\tau(t) = \gamma(s(t))$. Differentiating, we obtain

$$\frac{d\tau}{dt} = \frac{d\gamma}{ds}\frac{ds}{dt} = T\ \frac{ds}{dt}$$

$$\frac{d^2\tau}{dt^2} = \frac{d^2\gamma}{ds^2}\left(\frac{ds}{dt}\right)^2 + \frac{d\gamma}{ds}\frac{d^2s}{dt^2} = \kappa N\ \left(\frac{ds}{dt}\right)^2 + T\ \frac{d^2s}{dt^2}$$

Since we know $\tau(t)$, we can compute $\frac{d\tau}{dt}$. The first equation then allows us to compute T and $\frac{ds}{dt}$ because T is $\frac{d\tau}{dt}$ normalized to have length one and $\frac{ds}{dt}$ is the length of $\frac{d\tau}{dt}$.

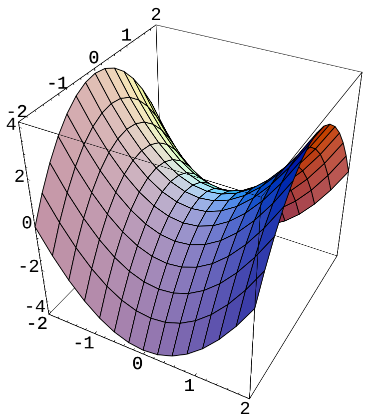
Put this information in the second equation. Since $\tau(t)$ is known, we can compute $\frac{d^2\tau}{dt^2}$ directly. We know everything in this equation except κ and N, so we can compute these. Once we know κ and N, we easily find the center of the osculating circle.

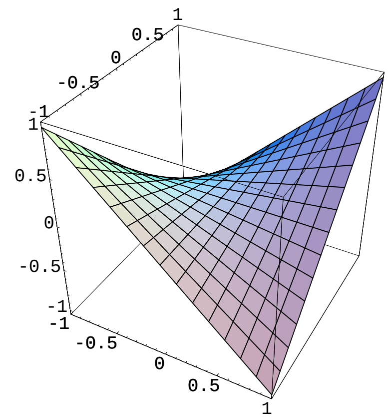
2 Surfaces

When we study surfaces, we always produce two-dimensional coordinates for the surface and do calculations in the coordinate system. So all hard work is two-dimensional rather than three dimensional.

We use one of two techniques. When the surface is given by $z = f(x, y)$, we use x and y as coordinates, so the two dimensional point (x, y) corresponds to the three dimensional point $(x, y, f(x, y))$. When the surface is more complicated, we produce a function $s(u, v) = (x(u, v), y(u, v), z(u, v))$ which maps the two-dimensional point (u, v) to the three dimensional point (x, y, z).

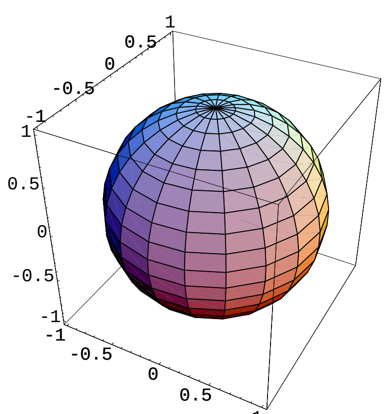
Exercise 1: Consider the functions $f(x, y) = y^2 - x^2$ and $g(x, y) = xy$. Convince yourself that the graphs have the following shapes. Notice that both are saddles. Notice that the g saddle is the f saddle rotated 45 degrees (up to a scaling factor).

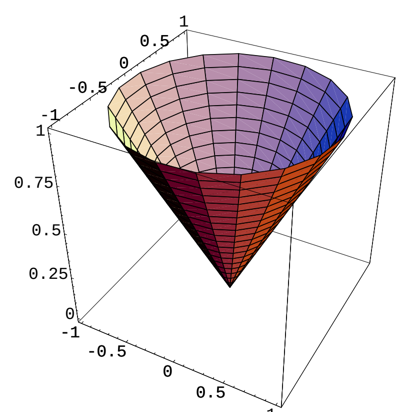




Exercise 2: Explain why $s(\varphi, \theta) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$ parameterizes the sphere as shown on the left below.

Exercise 3: Explain why $s(r, \theta) = (r\cos\theta, r\sin\theta, r)$ parameterizes the cone as shown on the right below.





Remark: On the surface, we have the standard Euclidean geometry method to measure distances, lengths of curves, and angles. Because the coordinates distort these quantities, we measure them in coordinates using the metric tensor g_{ij}. Here are the key facts:

1. If $z = f(x, y)$, then

$$g_{11} = 1 + \left(\frac{\partial f}{\partial x}\right)^2 \qquad g_{12} = \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} \qquad g_2 = 1 + \left(\frac{\partial f}{\partial y^2}\right)^2$$

2. If we have $s(u, v) = (x(u, v), y(u, v), z(u, v))$, then

$$g_{11} = \frac{\partial s}{\partial u} \cdot \frac{\partial s}{\partial u} \qquad g_{12} = \frac{\partial s}{\partial u} \cdot \frac{\partial s}{\partial v} \qquad g_{22} = \frac{\partial s}{\partial v} \cdot \frac{\partial s}{\partial v}$$

3. In the Poincare disk we take

$$g_{11} = \frac{2}{1 - x^2 - y^2} \qquad g_{12} = 0 \qquad g_{22} = \frac{2}{1 - x^2 - y^2}$$

   This metric does not come from a surface in three space.

4. A small segment from (u, v) to $(u + du, v + dv)$ has length

$$\sqrt{g_{11}du^2 + 2g_{12}du\ dv + g_{22}dv^2}$$

   This formula can be written in a more compact way if we let $g_{21} = g_{12}$ and write $u_1 = u$ and $u_2 = v$:

$$\sqrt{\sum g_{ij}u_iu_j}$$

5. Suppose $\gamma(t) = (u_1(t), u_2(t))$ for $a \le t \le b$ is a curve in the coordinate system. Then $s(u_1(t), u_2(t)) = (x(t), y(t), z(t))$ is the corresponding curve on the surface. The length of the curve on the surface is given by standard Euclidean geometry as

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

   This is exactly equal to the length in coordinates computed using the formula

$$\int_a^b \sqrt{\sum g_{ij}\ \frac{du_i}{dt}\frac{du_j}{dt}}\ dt$$

6. Suppose the surface is given by $z = f(x, y)$ and (A, B) is a vector in the coordinate system. The corresponding vector on the surface is

$$\left(A, B, \frac{\partial f}{\partial x}A + \frac{\partial f}{\partial y}B\right)$$

7. Suppose instead that the surface is given by $s(u,v) = (x(u,v), y(u,v), z(u,v))$. If (A, B) is a vector in the coordinate system, the corresponding vector on the surface is

$$\frac{\partial s}{\partial u}A + \frac{\partial s}{\partial v}B$$

8. Changing the notation slightly, suppose we have two vectors $A = (A_1, A_2)$ and $B = (B_1, B_2)$ in coordinates. Call the resulting vectors on the surface $U = (U_1, U_2, U_3)$ and $V = (V_1, V_2, V_3)$. Another way to interprete our length formulas is that we get the same result if we compute the length of A using the formula on the left below and the length of U using the formula on the right below:

$$\sqrt{\sum g_{ij} A_i A_j} \quad = \quad \sqrt{U_1^2 + U_2^2 + U_3^2}$$

9. Similarly, we get the same result if we compute the dot product of A and B using the formula on the left below and the dot product of U and V using the formula on the right below:

$$\langle A, B \rangle = \sum g_{ij} A_i B_j \quad = \quad U \cdot V = U_1 V_1 + U_2 V_2 + U_3 V_3$$

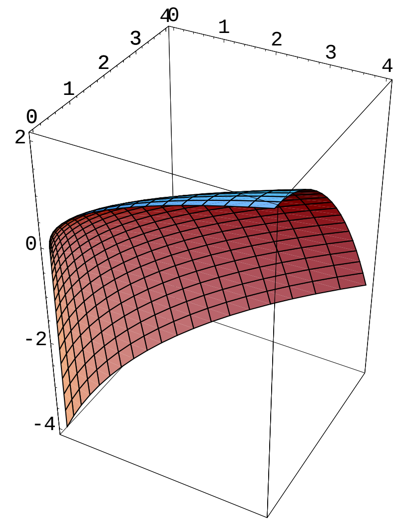
Remark: We can summarize these results very simply: lengths and angles on the surface can be computed using similar formulas in two-dimensional coordinates is we use g_{ij} rather than the standard Euclidean formulas.

Exercise 4: Consider the surface $z = y^2 - x^2$. Compute the g_{ij}. Let $A = (1, 2)$ and $B = (3, 4)$ be vectors at the point $(1, 7)$. Find the corresponding three-dimensional vectors U and V on the surface. Compute the length of U and the dot product of U and V using standard formulas. Compute the length of A and the dot product of A and B using our g_{ij} formulas. Show that you get the same result.

Exercise 5: Consider the cone $s(r, \theta) = (r \cos\theta, r \sin\theta, r)$. Compute the g_{ij}. Let $A = (1, 2)$ and $B = (3, 4)$ be vectors at the point $(1, \pi/4)$. Find the corresponding three-dimensional vectors U and V on the surface. Repeat exercise 4 in this case.

Exercise 6: Consider the surface $z = f(x, y) = xy$. Compute g_{ij}. Consider the parabola (t, t^2) for $0 \le t \le 1$ in the plane. Find the corresponding curve on the surface. Write down the integral which gives the length of this surface curve, but do not evaluate it. Write down the integral which gives the length of the original parabola, *but using the* g_{ij}. Show that the two integrals are equal.

Exercise 7: Repeat this calculation for the surface $s(u, v) = (u^2, uv, u - v^2)$.



Exercise 8: Consider the curves (t, t^2) and (t^2, t^3) in the plane. These curves meet at $(1, 1)$. Find the corresponding curves in three-space using the s of the previous example. These curves meet at $s(1, 1)$. Find the cosine of the angle between their tangents.

Now compute the same cosine using the two-dimensional forms (t, t^2) and (t^2, t^3) by taking two-dimensional tangents and computing dot products using the g_{ij} formula.

Remark: Finally, we have Christoffel symbols, geodesics, and all of that. Here are the key points:

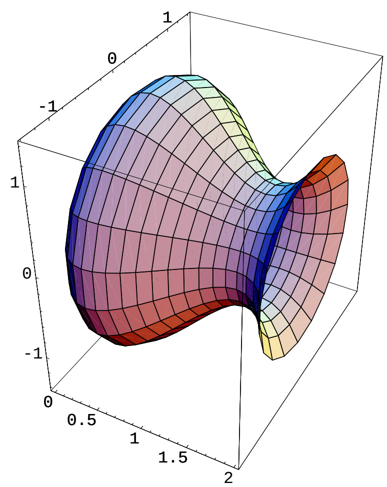
1. Consider curves which start at p and end at q. Instead of minimizing the length of such a curve, we can minimize the energy. A curve minimizes energy if and only if it minimizes length and is parameterized by arc length.
2. A calculus of variations argument then shows that if a curve minimizes energy, it solves the geodesic equation. I want you to know this argument. It is extremely unlikely that I will ask for a complete proof of the argument. But I could sketch part of the argument and then ask you to give the next step.
3. The ultimate result is that geodesics satisfy the equation

$$\frac{d^2\gamma_i}{dt^2} + \sum_{jk} \Gamma^i_{jk} \frac{d\gamma_j}{dt} \frac{d\gamma_k}{dt} = 0$$

4. Here

$$\Gamma^i_{jk} = \frac{1}{2} \sum g^{-1}_{il} \left(\frac{\partial g_{li}}{\partial x_j} + \frac{\partial g_{lj}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_l} \right)$$

Exercise: Consider a surface of revolution obtained by revolving $y = f(x)$ about the x-axis. The surface can be parameterized by $s(x, \theta) = (x, f(x)\cos\theta, f(x)\sin\theta)$. Compute the g_{ij}. Compute the Γ^i_{jk}. Write down the two differential equations. Solve the second one, and then describe the geodesics in words. When is a circle around the surface a geodesic?



Remark: If you can do this, you've got it made!