Review 2

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1 Curves

Here are the key facts you'll need for calculations:

- If $\gamma(t)$ is a curve, then its length from t_0 to t is $s(t) = \int_{t_0}^{t} ||\gamma'(t)||\ dt$
- If $\gamma(u)$ is parameterized by arc length, then κ, T, N, B can be computed using the formulas

$$T = \frac{d\gamma}{du} \qquad \frac{dT}{du} = \kappa N \qquad B = T \times N$$

- If $\gamma(t)$ is not parameterized by arc length, then κ, T, N, B can be computed using the formulas

$$T = \frac{\gamma'(t)}{||\gamma'(t)||} \qquad \gamma''(t) = T\frac{d^2 s}{dt^2} + \kappa N \left(\frac{ds}{dt}\right)^2 \qquad \kappa = \frac{||\gamma' \times \gamma''||}{||\gamma'||^3} \qquad B = T \times N$$

In the second formula on the last line, notice that $\frac{ds}{dt} = \gamma'(t)$. Consequently you can compute every term in this formula except κN and thus you can use it to find κ and N.

I will not ask you to calculate τ.

Example Let $\gamma(t)$ be the curve $y = x^3$. Compute κ, T, and N at $x = 1$.

Example Let $\gamma(t) = (t, t^2, t^3)$. Compute κ, T, and N at $x = 1$.

2 Theory of Curves

You need to know

- The statement of the Frenet-Serret formulas
- The proof of the Frenet-Serret formulas
- The proof of the theorem that if $\gamma(u)$ and $\sigma(u)$ are curves with nowhere vanishing curvature and the same $\kappa(u)$ and $\tau(u)$, then it is possible to rotate and translate γ so it lies on top of σ.
- How to convert the Frenet-Serret formulas to a recursive algorithm to find $\gamma(u)$ on a computer if you know $\kappa(u)$ and $\tau(u)$.

  Remark: For the first item, see page 10 of the notes. For the second item, see pages 8 - 9 of the notes. For the third item, see pages 12 - 14 of the notes, and for the final item see page 14 of the notes.

3 Parameterized Surfaces

In this section, you need to know

- A surface is given by $z = f(x, y)$ or $s(u, v) = (x(u, v), y(u, v), z(u, v))$. The first form can be converted to the second form by writing $s(x, y) = (x, y, f(x, y))$.

- A normal to the surface is given by $N = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right)$ or $N = \frac{\partial s}{\partial u} \times \frac{\partial s}{\partial v}$

- A unit normal is given by $n = \frac{N}{||N||}$.

- The g_{ij} are given by

$$g_{11} = 1 + \left(\frac{\partial f}{\partial x}\right)^2 \qquad g_{12} = \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} \qquad g_{22} = 1 + \left(\frac{\partial f}{\partial y}\right)^2$$

or

$$g_{11} = \frac{\partial s}{\partial u} \cdot \frac{\partial s}{\partial u} \qquad g_{12} = \frac{\partial s}{\partial u} \cdot \frac{\partial s}{\partial v} \qquad g_{22} = \frac{\partial s}{\partial v} \cdot \frac{\partial s}{\partial v}$$

- Tangent vectors in the coordinate plane correspond to tangent vectors in R^3 by:

$$X = (a, b) \leftrightarrow \tilde{X} = \left(a, b, a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y}\right)$$

or

$$X = (a, b) \leftrightarrow \tilde{X} = a\frac{\partial s}{\partial u} + b\frac{\partial s}{\partial v}$$

- $< X, Y >= \tilde{X} \cdot \tilde{Y} = \sum g_{ij} X_i Y_j$

- $||X||_{\text{fancy}} = ||\tilde{X}||_{\text{ordinary}} = \sum g_{ij} X_i X_j$

- The fancy length of a curve $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ is

$$\int \sqrt{\sum g_{ij}(\gamma(t)) \frac{d\gamma_i}{dt} \frac{g\gamma_j}{dt}}$$

Example Parameterize the top half of the sphere by $z = f(x, y)$. Parameterize the entire sphere using spherical coordinates (ϕ, θ). Parameterize the saddle $z = xy$ using rectangular coordinates. Parameterize this saddle using polar coordinates.

Example Let $X = (2, 3)$ and $Y = (4, 5)$ at the point $(1, 1, 1)$ on the saddle. Compute g_{ij} and compute $< X, Y >$ and compute $||X||$. Compute $\tilde{X}$ and $\tilde{Y}$ and compute the dot product of these three dimensional vectors and the ordinary length of $\tilde{X}$.

4 Geodesics

You need to know

- $$\Gamma^k_{ij} = \frac{1}{2} \sum_l \left(g^{-1}\right)_{kl} \left(\frac{\partial g_{il}}{\partial x_j} + \frac{\partial g_{jl}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_l} \right)$$

- $$\frac{d^2\gamma_k}{dt^2} + \sum_{ij} \Gamma^k_{ij} \frac{d\gamma_i}{dt} \frac{d\gamma_j}{dt} = 0$$

- What geodesics on a sphere look like, and why

- What geodesics on a surface of revolution look like pictorially, though not the details of the calculation

Example: Consider the paraboloid $z = x^2 + y^2$. Parameterize in polar coordinates. Compute the g_{ij} and the Γ^k_{ij}. Write down the differential equations satisfied by geodesics. Show that curves with θ constant are geodesics if we parameterize correctly (i.e., show that only one equation remains and it is equivalent to the statement that the radial curve has constant speed). Show that curves with constant r are not geodesics.

5 Proof of the Geodesic Equation

You should know

- The definition of energy, and the statement that minimizing energy is equivalent to miniimizing length plus traveling at constant speed, but not necessary the proof of this statement.

- The proof that a curve of minimal energy satisfies the geodesic equation, as it unfolded in the first midterm.

6 Differentiating Functions and Vectors

You need to know

- To compute $X(g)$, where g is a function $g(x, y, z)$, we pull the function back to coordinates by writing $g(s(u, v))$ or $g(x, y, f(x, y))$ and then differentiate using the standard formulas from calculus. For example, if $X = (2, 3)$ we change notation to

$$X = 2\frac{\partial}{\partial u} + 3\frac{\partial}{\partial v}$$

  and then write

$$X(g) = 2\frac{\partial g}{\partial u} + 3\frac{\partial g}{\partial v}$$

- To differentiate an arbitrary three-dimensional vector $U = (U_1, U_2, U_3)$ on the surface in the direction X, we differentiate each component separately.

- The answer will be another three-dimensional vector $V = X(U)$. To decompose

$$V = \lambda n + Y$$

  where λ is a constant, n is the unit normal, and Y is tangent, we dot both sides with respect to n to obtain $\lambda = V \cdot n$ and then write $Y = V - \lambda n$

- We then have

$$\begin{array}{rcrcl} X(n) & = & & & B(X) \\ X(Y) & = & b(X, Y)n & + & \nabla_X Y \end{array}$$

- It is now possible to write down a matrix for B as follows. Choose $X = \frac{\partial}{\partial u}$ and $Y = \frac{\partial}{\partial v}$. Compute n; you will get a three dimensional vector which is a function of u and v. Compute $X(n)$. The answer will be a three dimensional vector, but it will actually be tangent to the surface. So it can be written $a\frac{\partial s}{\partial u} + c\frac{\partial s}{\partial v}$. Write a and c in the first column of the matrix for B. Repeat with $Y(n)$, calling the resulting coefficients b and d. Then

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

- The eigenvalues of B can be computed by solving $\det(\lambda I - B) = 0$ for λ. Call these eigenvalues $-\kappa_1$ and $-\kappa_2$. These numbers are the principal curvatures of the surface at the point (u, v).

Example: Let $g = xy + z^2$ on the surface $z = x^2 + y^2$. Let $X = (2, 3)$ at $(1, 1, 2)$. Compute $X(g)$.

Example: Let $U = (y, x, z)$ on the same surface. Compute $X(U)$. Then decompose this vector into a normal and tangential component.

Example: Let $g = x^2 + y^2$. Parameterize this surface in polar coordinates. Compute the matrix B. Then compute the principal curvatures at arbitrary points (r, θ). By symmetry, the answer should not depend on θ.