Christoffel Symbols: Only derivatives with respect to the second of (θ, ϕ) are nonzero, and only when differentiating $g_{11} = \sin^2 \phi$. So the only nonzero terms are

$$\Gamma^2_{11} = \frac{1}{2}\frac{1}{g_{22}}\left(-\frac{\partial g_{11}}{\partial \phi}\right) = -\sin\phi\cos\phi$$

$$\Gamma^1_{12} = \frac{1}{2}\frac{1}{g_{11}}\left(\frac{\partial g_{11}}{\partial \phi}\right) = \frac{\cos\phi}{\sin\phi}$$

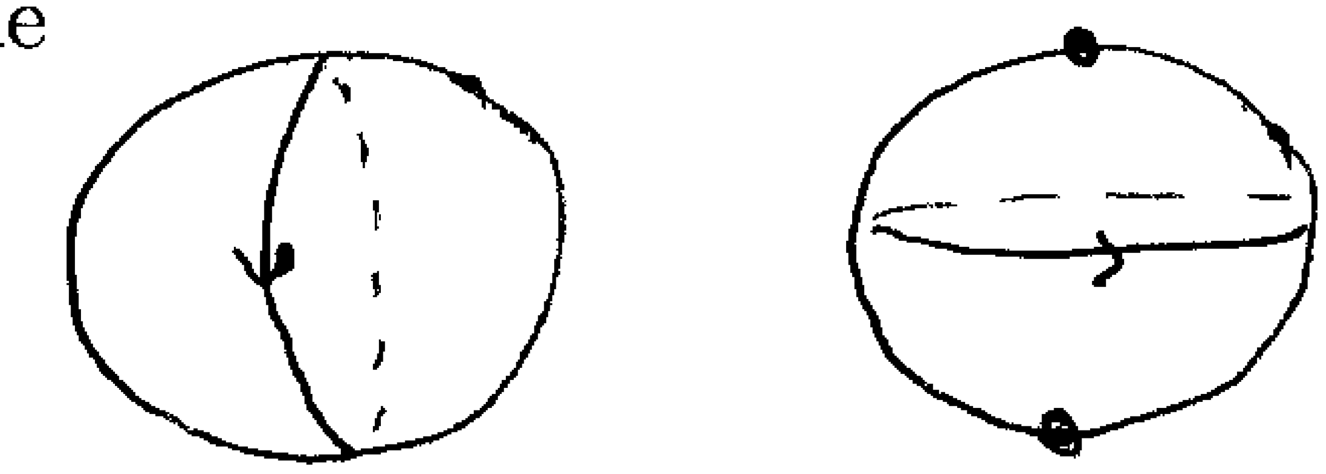
Geodesic Equations: In general these equations are

$$\frac{d^2\gamma_k}{dt^2} + \sum_{ij} \Gamma^k_{ij} \frac{d\gamma_i}{dt}\frac{d\gamma_j}{dt} = 0$$

In the case of the sphere these equations become

$$\frac{d^2\theta}{dt^2} + \frac{2\cos\phi}{\sin\phi}\frac{d\theta}{dt}\frac{d\phi}{dt} = 0$$

$$\frac{d^2\phi}{dt^2} - \sin\phi\cos\phi\left(\frac{d\theta}{dt}\right)^2 = 0$$



Notice that when θ is constant, the equations are solved by $\phi(t) = at + b$, but if ϕ is constant, the equations are solved by $\theta(t) = at + b$ only when $\sin\phi\cos\phi = 0$, and thus only at the equator and the north and south poles.

Intrinsic Tangential Calculation: We can also compute $\nabla_X Y$ intrinsically using the Christoffel symbols. In this calculation,

$$\frac{\partial}{\partial u_i}(Y_1, Y_2) = \left(\frac{\partial Y_1}{\partial u_i}, \frac{\partial Y_2}{\partial u_i}\right) + \left(\Gamma^1_{i1}Y_1 + \Gamma^1_{i2}Y_2, \Gamma^2_{i1}Y_1 + \Gamma^2_{i2}Y_2\right)$$

When we use this formula to differentiate $e_1 = (1, 0)$ and $e_2 = (0, 1)$, the partial derivative portion vanishes and only the Christoffel portion remains:

$$\nabla_{e_1} e_1 = -\sin\phi\cos\phi \; e_2$$

$$\nabla_{e_1} e_2 = \frac{\cos\phi}{\sin\phi} \; e_1$$

$$\nabla_{e_2} e_1 = \frac{\cos\phi}{\sin\phi} \; e_1$$

$$\nabla_{e_2} e_2 = 0$$