Brief Article

The Author

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Folks,

Here is the answer to the geodesic question on page four of the review sheet.

The parameterization is $s(r,\theta) = (r\cos\theta, r\sin\theta, r^2)$. Then

$$\frac{\partial s}{\partial r} = (\cos\theta, \sin\theta, 2r)$$

$$\frac{\partial s}{\partial \theta} = (-r\sin\theta, r\cos\theta, 0)$$

So

$$g_{11} = 1 + 4r^2$$

$$g_{12} = 0$$

$$g_{22} = r^2$$

The nonzero Christoffel symbols are then

$$\Gamma^1_{11} = \frac{1}{2}\frac{1}{1+4r^2}8r = \frac{4r}{1+4r^2}$$

$$\Gamma^1_{22} = \frac{1}{2}\frac{1}{1+4r^2}(-2r) = \frac{-r}{1+4r^2}$$

$$\Gamma^2_{12} = \frac{1}{2}\frac{1}{r^2}(2r) = \frac{1}{r}$$

So the differential equations are

$$\frac{d^2r}{dt^2} + \frac{4r}{1+4r^2}\left(\frac{dr}{dt}\right)^2 - \frac{r}{1+4r^2}\left(\frac{d\theta}{dt}\right)^2 = 0$$

$$\frac{d^2\theta}{dt^2} + \frac{2}{r}\frac{dr}{dt}\frac{d\theta}{dt} = 0$$

If r is constant, these equations become

$$-\frac{r}{1+4r^2}\left(\frac{d\theta}{dt}\right)^2 = 0$$

$$\frac{d^2\theta}{dt^2} = 0$$

According to the second equation, $\theta(t) = at + b$. If $a = 0$, this curve remains at a fixed point, which is not very interesting. Otherwise the first equation becomes

$$\frac{-r}{1+4r^2}a^2 = 0$$

and this only holds if $r = 0$. But then again we are at the origin and our geodesic never moves.

If θ is constant, these equations become

$$\frac{d^2r}{dt^2} + \frac{4r}{1+4r^2}\left(\frac{dr}{dt}\right)^2 = 0$$

This equation just says that the curve is traced at constant speed. Indeed, the distance traveled when we increase r by dr is $\sqrt{(dr)^2 + (2rdr)^2} = \sqrt{1+4r^2}dr.$, so the speed is $\sqrt{1+4r^2}\frac{dr}{dt}$. This is constant exactly when $\frac{d}{dt}\sqrt{1+4r^2}\frac{dr}{dt} = 0$. So

$$\frac{1}{2}\frac{8r}{\sqrt{1+4r^2}}\left(\frac{dr}{dt}\right)^2 + \sqrt{1+4r^2}\frac{d^2r}{dt^2} = 0$$

or equivalently

$$\frac{d^2r}{dt^2} + \frac{4r}{1+4r^2}\left(\frac{dr}{dt}\right)^2 = 0.$$

Dick