Final Exercise

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Here is the answer to the final exercise on the review sheet.

Let me do the calculation. The parameterization is $s(r, \theta) = (r\cos\theta, r\sin\theta, r^2)$. Then

$$\frac{\partial s}{\partial r} = (\cos\theta, \sin\theta, 2r)$$

$$\frac{\partial s}{\partial \theta} = (-r\sin\theta, r\cos\theta, 0)$$

The cross product of these vectors is

$$N = (-2r^2\cos\theta, -2r^2\sin\theta, r)$$

and n is this vector divided by its length $\sqrt{4r^4 + r^2} = r\sqrt{1 + 4r^2}$, which is

$$n = \frac{(-2r\cos\theta, -2r\sin\theta, 1)}{\sqrt{1 + 4r^2}}$$

Let us take as our basis vectors $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$. Recall that (a, b) corresponds to the three dimensional vector $a\frac{\partial s}{\partial r} + b\frac{\partial s}{\partial \theta}$. So $(1, 0) = \frac{\partial}{\partial r}$ corresponds to the three dimensional vector $(\cos\theta, \sin\theta, 2r)$ and $(0, 1) = \frac{\partial}{\partial \theta}$ corresponds to the three dimensional vector $(-r\sin\theta, r\cos\theta, 0)$.

Now the main calculation. We have

$$\frac{\partial}{\partial r} n = \frac{(-2\cos\theta, -2\sin\theta, -4r)}{(1 + 4r^2)^{3/2}} = \frac{-2}{(1 + 4r^2)^{3/2}} \frac{\partial}{\partial r}$$

where I have the details on a piece of paper but won't write them here. An easier calculation gives

$$\frac{\partial}{\partial \theta} n = \frac{(2r\sin\theta, -2r\cos\theta, 0)}{\sqrt{1 + 4r^2}} = \frac{-2}{\sqrt{1 + 4r^2}} \frac{\partial}{\partial \theta}$$

Therefore the matrix B is

$$B = \begin{pmatrix} \frac{-2}{(1+4r^2)^{3/2}} & 0 \\ 0 & \frac{-2}{\sqrt{1+4r^2}} \end{pmatrix}$$

Notice that this matrix does not depend on θ. Since the matrix is diagonal, the eigenvalues are the diagonal entries. So

$$\kappa_1 = \frac{2}{(1+4r^2)^{3/2}}$$

$$\kappa_2 = \frac{2}{\sqrt{1+4r^2}}$$