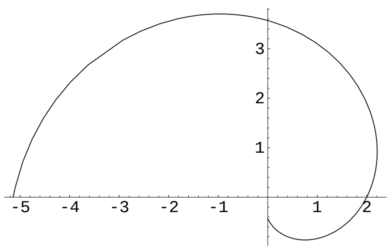
Mathematics 433/533 Midterm

April 29, 2005

Name ____

1. (12) The curve $\gamma(t) = ((t+2)\cos t, (t+2)\sin t)$ goes through the point $(2,0)$ when $t = 0$.

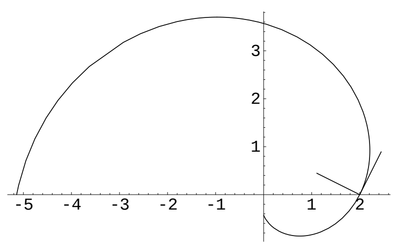


(a) Find the unit tangent vector T to the curve at $t = 0$.

Answer: The derivative of the curve is $\gamma'(t) = (\cos t - (t+2)\sin t, \sin t + (t+2)\cos t)$ and so $\gamma'(0) = (1,2)$. This vector has length $\sqrt{1^2 + 2^2} = \sqrt{5}$, so $T = \frac{1}{\sqrt{5}}(1,2)$.

(b) Find the unit normal N at $t = 0$. You can do this by looking at the above T and drawing a picture, without much calculation.

Answer: Below is a picture of the curve, T, and N. Notice that N goes left by the same amount that T goes up, and N goes up by the same amount that T goes right. So $N = \frac{1}{\sqrt{5}}(-2,1)$. As a check, this vector has length one, and its dot product with T is zero.



(c) Find the curvature κ at $t = 0$.

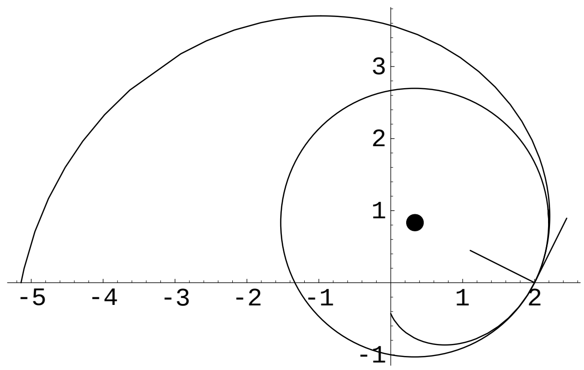
Answer: The second derivative of the curve is $\gamma''(t) = (-2\sin t - (t+2)\cos t, 2\cos t - (t+2)\sin t)$ and so $\gamma''(0) = (-2,2)$. Thinking of the curve in three dimensions, we have

$$\kappa = \frac{||\gamma' \times \gamma''||}{||\gamma'||^3} = \frac{||(1,2,0)\times(-2,2,0)||}{\sqrt{5}^3} = \frac{6}{\sqrt{5}^3} = \left(\frac{1}{3}, \frac{5}{6}\right)$$

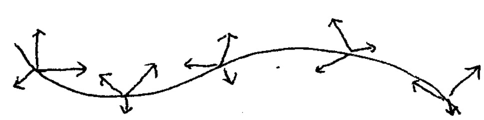
(d) Find the center of the osculating circle at $t = 0$.

Answer: The center is

$$\gamma(0) + \frac{1}{\kappa}T = (2,0) + \frac{\sqrt{5}^3}{6}\frac{1}{\sqrt{5}}(-2,1) = (2,0) + \frac{5}{6}(-2,1) = \left(\frac{1}{3}, \frac{5}{6}\right)$$



2. (12) Let $\gamma(t)$ be a parameterized curve. Suppose $X_1(t), X_2(t)$, and $X_3(t)$ are three orthonormal vectors attached to $\gamma(t)$ for each t. See the picture below. These vectors need not be the vectors T, N, and B.



Write

$$\frac{dX_i}{dt} = \sum_j a_{ij} X_j$$

and let $A = (a_{ij})$ be the resulting matrix. Prove that $A^T = -A$.

Answer: Since the X_i are orthonormal, $X_i \cdot X_j = \delta_{ij}$ where this delta is one if $i = j$ and zero otherwise. Differentiate both sides with respect to t:

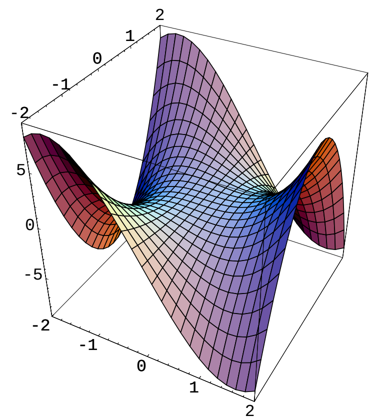
$$\frac{dX_i}{dt} \cdot X_j + X_i \cdot \frac{dX_j}{dt} = \frac{d}{dt}\delta_{ij} = 0$$

and so

$$\left(\sum_k a_{ik} X_k\right) \cdot X_j + X_i \cdot \left(\sum_k a_{jk} X_k\right) = 0$$

In the dot product at the extreme left, the term $X_k \cdot X_j$ is zero unless $k = j$, and one in this special case. So this expression is just a_{ij}. Similarly, the second half of the expression on the left is a_{ji}. So $a_{ij} + a_{ji} = 0$, or $A + A^T = 0$, so $A^T = -A$.

3. (12) The surface $z = x^3 - 2xy^2$ is called a *monkey saddle*.



(a) Find a normal vector at $(1, 1, -1)$.

Answer: If a surface is given by $z = f(x, y)$, then each normal to the surface has the form $n = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right)$. In our case,

$$n = \left(-(3x^2 - 2y^2), -(-4xy), 1\right) = (-1, 4, 1)$$

(b) Compute the g_{ij} at $(1, 1)$.

Answer:

$$g_{11} = 1 + \left(\frac{\partial f}{\partial x}\right)^2 = 1 + 1^2 = 2$$

$$g_{12} = \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} = (1)(-4) = -4$$

$$g_{22} = 1 + \left(\frac{\partial f}{\partial y}\right)^2 = 1 + (-4)^2 = 17$$

(c) Let $X = (1, 1)$ and $Y = (1, 2)$ be vectors at $(1, 1)$. Find the fancy length $||X||$ and the fancy inner product $< X, Y >$.

Answer:

$$||X||^2 = g_{11}1^2 + 2g_{12}(1)(1) + g_{22}1^2 = 2 - 8 + 17 = 11, \text{ so } ||X|| = \sqrt{11}$$

$$< X, Y >= g_{11}1^2 + g_{12}\left((1)(2) + (1)(1)\right) + g_{22}(1)(2) = 2 - 12 + 34 = 24$$

(d) Find the corresponding tangent vectors $\tilde{X}$ and $\tilde{Y}$ on the surface, and compute their dot product. You should get a number which you have already obtained. Why?

Answer: In general if $X = (a, b)$ then $\tilde{X} = \left(a, b, a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y}\right)$, so

$$\tilde{X} = (1, 1, 1 - 4) = (1, 1, -3)$$

$$\tilde{Y} = (1, 2, 1 - 8) = (1, 2, -7)$$

and the dot product of these vectors is $1 + 2 + 21 = 24$. This holds because $< X, Y >= \tilde{X} \cdot \tilde{Y}$ by definition.

4. (12) In the derivation of the geodesic equation, we assumed that a curve $\gamma(t)$ for $a \le t \le b$ minimizes energy among all curves from $\gamma(a)$ to $\gamma(b)$. If $\delta(t)$ is a variation of the curve, we considered a family of new curves

$$\gamma_u(t) = \gamma(t) + u\delta(t)$$

The energy of these new curves is

$$\mathcal{E}_u = \int_a^b \sum_{ij} g_{ij}(\gamma(t) + u\delta(t)) \frac{\partial}{\partial t}(\gamma_i(t) + u\delta_i(t)) \frac{\partial}{\partial t}(\gamma_j(t) + u\delta_j(t))\ dt.$$

Since $\mathcal{E}_u$ is minimal when $u = 0$, the derivative of $\mathcal{E}_u$ with respect to u must vanish at $u = 0$. Carry out the calculation of this derivative until you obtain an expression of the form below involving $\delta(t)$ but not its derivative. You need not simplify once you get this expression.

$$\int_a^b \sum_k (\text{complicated expression in } t)\, \delta_k(t)\ dt$$

When the calculation is done, you'll have an expression with an integral sign. End by explaining *in words* why the expression with an integral sign can be converted to a pure differential equation with no integral sign and no δ.

Answer: Since the integrand is a product of three terms, the derivative of the integral with respect to u is an integral of three terms:

$$\int_a^b \sum_{ij} \left[\sum_k \frac{\partial g_{ij}}{\partial x_k} \delta_k \frac{d\gamma_i}{dt} \frac{d\gamma_j}{dt} + g_{ij}\delta_i' \frac{d\gamma_j}{dt} + g_{ij} \frac{d\gamma_i}{dt} \delta_j' \right]\ dt = 0$$

We integrate the second and third terms by parts, noticing that the boundary term vanishes at the endpoints because $\delta(a) = \delta(b) = 0$. This gives

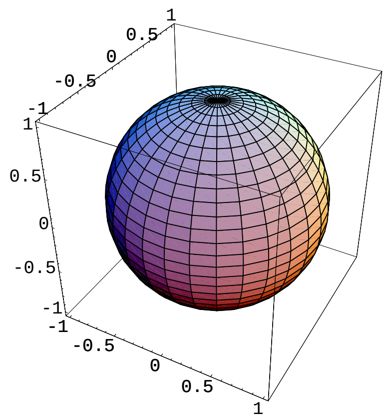
$$\int_a^b \sum_{ij} \left[\sum_k \frac{\partial g_{ij}}{\partial x_k} \delta_k \frac{d\gamma_i}{dt} \frac{d\gamma_j}{dt} - \frac{d}{dt}\left(g_{ij} \frac{d\gamma_j}{dt} \right) \delta_i - \frac{d}{dt}\left(g_{ij} \frac{d\gamma_i}{dt} \right) \delta_j \right]\ dt = 0$$

In the second term, change i to k; in the third term, change j to k. This gives

$$\int_a^b \sum_k \left[\sum_{ij} \frac{\partial g_{ij}}{\partial x_k} \frac{d\gamma_i}{dt} \frac{d\gamma_j}{dt} - \frac{d}{dt}\left(\sum_j g_{kj} \frac{d\gamma_j}{dt} \right) - \frac{d}{dt}\left(\sum_i g_{ik} \frac{d\gamma_i}{dt} \right) \right] \delta_k\ dt = 0$$

This integral must vanish for any choice of δ_k, which can only happen if the integrand inside the large square brackets is identically zero. QED.

5. (12) Consider standard spherical coordinates $(\phi, \theta) \to (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$ on the sphere. Recall that ϕ is the angle down from the north pole and θ is the angle in the xy-plane. On maps, ϕ gives latitude and θ gives longitude.



(a) Compute g_{11}, g_{12}, g_{22} for this parameterization.

Answer:

$$\frac{\partial s}{\partial \phi} = (\cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi)$$

$$\frac{\partial s}{\partial \theta} = (-\sin\phi\sin\theta, \sin\phi\cos\theta, 0)$$

$$g_{11} = \frac{\partial s}{\partial \phi} \cdot \frac{\partial s}{\partial \phi} = 1$$

$$g_{12} = \frac{\partial s}{\partial \phi} \cdot \frac{\partial s}{\partial \theta} = 0$$

$$g_{22} = \frac{\partial s}{\partial \theta} \cdot \frac{\partial s}{\partial \theta} = \sin^2\phi$$

(b) Compute Γ^1_{22}. Please give details; don't just give the answer.

Answer: $\Gamma^1_{22} = \frac{1}{2}(g^{-1})_{11}\left(\frac{\partial g_{12}}{\partial x_2} + \frac{\partial g_{12}}{\partial x_2} - \frac{\partial g_{22}}{\partial x_1}\right) = \frac{1}{2}\frac{1}{1}(-2\sin\phi\cos\phi) = -\sin\phi\cos\phi$

(c) As a check of your previous answer, $\Gamma^1_{22} = -\sin\phi\cos\phi$ and $\Gamma^2_{12} = \frac{\cos\phi}{\sin\phi}$ and all other Christoffel symbols are zero. Using this fact, write the differential equations which geodesics must satisfy. You can answer this and remaining questions even if you made mistakes in a) and b).

Answer:

$$\frac{d^2\phi}{dt^2} - \sin\phi\cos\phi\left(\frac{d\theta}{dt}\right)^2 = 0$$

$$\frac{d^2\theta}{dt^2} + \frac{2\cos\phi}{\sin\phi}\frac{d\phi}{dt}\frac{d\theta}{dt} = 0$$

(d) What do the differential equations say about geodesics with θ constant? Draw one of these geodesics on the above picture.

Answer: If θ is constant, $\frac{d\theta}{dt} = 0$, so the second equation is automatically true and the first equation reads $\frac{d^2\phi}{dt^2} = 0$. So ϕ increases at a constant rate. The resulting curve is a great circle that goes from the north pole to the south pole along some constant longitude. This curve is, of course, a great circle.

(e) What do the differential equations say about geodesics with ϕ constant? Explain this result in the picture.

Answer: In this case the equations reduce to

$$-\sin\phi\cos\phi\left(\frac{d\theta}{dt}\right)^2 = 0$$

$$\frac{d^2\theta}{dt^2} = 0$$

If $\frac{d\theta}{dt} = 0$, the curve is just a constant point, which is not very interesting. Otherwise $\sin\phi\cos\phi = 0$, and so $\phi = 0, \frac{\pi}{2}$, or π. The second equation just says that θ increases linearly. So the solutions are constant points at the north and south poles, or a circle around the equator. Notice that lines of constant latitude are usually not great circles; indeed the only such great circle is the line around the equator.