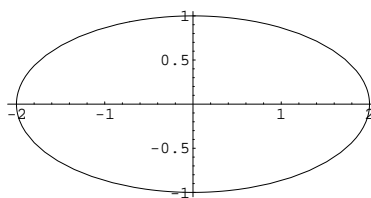


# Mathematics 433/533 Midterm 2

May 13, 2005

Name \_\_\_\_\_

1. (12) Consider the ellipse  $x^2 + 4y^2 = 4$ . This ellipse can be parameterized by  $\gamma(t) = (2 \cos t, \sin t)$ ; this is *not* a parameterization by arc length.



- (a) Find the unit tangent vector  $T$  to the curve at  $t = \frac{\pi}{4}$ .

**Answer:** We have  $\gamma' = (-2 \sin t, \cos t)$ , and at  $t = \frac{\pi}{4}$  this equals  $\left(-\frac{2}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}(-2, 1)$ . So the direction of the normal is  $(-2, 1)$ . Normalizing to get unit length, we have  $\frac{1}{\sqrt{5}}(-2, 1)$ .

- (b) Find the curvature  $\kappa$  at  $t = 0$  when  $\gamma$  is at the point  $(2, 0)$ .

**Answer:** We have  $\gamma' \times \gamma'' = (-2 \sin t, \cos t, 0) \times (-2 \cos t, -\sin t, 0) = (0, 0, 2)$ . The length of  $\gamma'$  is  $\sqrt{4 \sin^2 t + \cos^2 t}$ ; at  $t = 0$  this equals 1. So  $\kappa = 2$ .

- (c) Find the center of the osculating circle at  $(2, 0)$ .

**Answer:**  $\gamma(0) + \frac{1}{\kappa}N = (2, 0) + \frac{1}{2}(-1, 0) = \left(\frac{3}{2}, 0\right)$ .

2. (12) Consider the surface obtained by rotating the curve  $y = x^2$  about the  $x$ -axis. This surface can be parameterized by  $s(x, \theta) = (x, x^2 \cos \theta, x^2 \sin \theta)$  and then the unit normal is

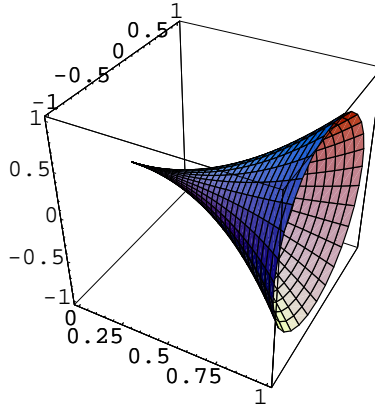
$$n = \frac{(2x, -\cos \theta, -\sin \theta)}{\sqrt{1 + 4x^2}}.$$

- (a) Let  $X = (1, 0) = \frac{\partial}{\partial x}$  and  $Y = (0, 1) = \frac{\partial}{\partial \theta}$ . Compute  $Y(n)$ . Show that your answer is a multiple of  $\tilde{Y}$ .

**Answer:**

$$\frac{\partial}{\partial \theta} \frac{(2x, -\cos \theta, -\sin \theta)}{\sqrt{1 + 4x^2}} = \frac{(0, \sin \theta, -\cos \theta)}{\sqrt{1 + 4x^2}}$$

Notice that  $X \leftrightarrow \tilde{X} = \frac{\partial s}{\partial \theta} = (0, -x^2 \sin \theta, x^2 \cos \theta)$ . So  $\frac{\partial}{\partial \theta} n = -\frac{1}{x^2 \sqrt{1 + 4x^2}} \tilde{Y}$ .



- (b) It can be shown that  $X(n) = \frac{2}{(1+4x^2)^{3/2}} \tilde{X}$ . Using this result and your calculation from the previous question, write down the matrix for  $B$ .

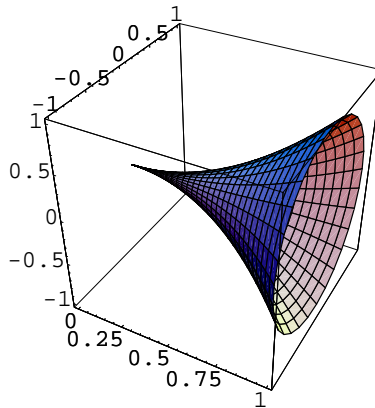
**Answer:**

$$B = \begin{pmatrix} -\frac{1}{x^2\sqrt{1+4x^2}} & 0 \\ 0 & \frac{2}{(1+4x^2)^{3/2}} \end{pmatrix}$$

- (c) What are  $\kappa_1$  and  $\kappa_2$ ? Using just these formulas for  $\kappa_1$  and  $\kappa_2$ , explain why the surface must look like a saddle locally, as the picture suggests.

**Answer:** So  $\kappa_1 = \frac{1}{x^2\sqrt{1+4x^2}}$  and  $\kappa_2 = -\frac{2}{(1+4x^2)^{3/2}}$ . One of these expressions is positive and one is negative, so the surface is locally a saddle.

3. (12) Consider once more the surface parameterized by  $s(x, \theta) = (x, x^2 \cos \theta, x^2 \sin \theta)$ .



- (a) Compute the  $g_{ij}$ .

**Answer:** We have  $\frac{\partial s}{\partial x} = (1, 2x \cos \theta, 2x \sin \theta)$  and  $\frac{\partial s}{\partial \theta} = (0, -x^2 \sin \theta, x^2 \cos \theta)$ , so

$$g_{11} = 1 + 4x^2$$

$$g_{12} = 0$$

$$g_{22} = x^4$$

(b) Compute  $\Gamma_{11}^1$

**Answer:**  $\Gamma_{11}^1 = \frac{1}{2} \frac{1}{1+4x^2} 8x = \frac{4x}{1+4x^2}$ .

(c) Compute the fancy length of the vector  $(1, 1)$  at an arbitrary point  $(x, \theta)$ .

**Answer:**  $g_{11} + 2g_{12} + g_{22} = 1 + 4x^2 + x^4$ .

(d) Draw the curve  $\gamma(t) = (t, t)$  for  $0 \leq t \leq 2\pi$  on the picture of the surface above. Then write down an integral giving the length of this curve. Please base your integral on fancy inner products. You need not evaluate the integral.

**Answer:**  $\int_0^{2\pi} \|\gamma'\|_{\text{fancy}} dt = \int_0^{2\pi} \sqrt{1 + 4t^2 + t^4} dt$

4. (12) In the derivation of the geodesic equation, we assumed that a curve  $\gamma(t)$  for  $a \leq t \leq b$  minimizes energy among all curves from  $\gamma(a)$  to  $\gamma(b)$ . If  $\delta(t)$  is a variation of the curve, we considered a family of new curves

$$\gamma_u(t) = \gamma(t) + u\delta(t)$$

The energy of these new curves is

$$\mathcal{E}_u = \int_a^b \sum_{ij} g_{ij}(\gamma(t) + u\delta(t)) \frac{\partial}{\partial t}(\gamma_i(t) + u\delta_i(t)) \frac{\partial}{\partial t}(\gamma_j(t) + u\delta_j(t)) dt.$$

Since  $\mathcal{E}_u$  is minimal when  $u = 0$ , the derivative of  $\mathcal{E}_u$  with respect to  $u$  must vanish at  $u = 0$ . Carry out the calculation of this derivative until you obtain an expression of the form below involving  $\delta(t)$  but not its derivative. You need not simplify once you get this expression.

$$\int_a^b \sum_k (\text{complicated expression in } t) \delta_k(t) dt$$

**Answer:** Since the integrand is a product of three terms, the derivative of the integral with respect to  $u$  is an integral of three terms:

$$\int_a^b \sum_{ij} \left[ \sum_k \frac{\partial g_{ij}}{\partial x_k} \delta_k \frac{d\gamma_i}{dt} \frac{d\gamma_j}{dt} + g_{ij} \delta'_i \frac{d\gamma_j}{dt} + g_{ij} \frac{d\gamma_i}{dt} \delta'_j \right] dt = 0$$

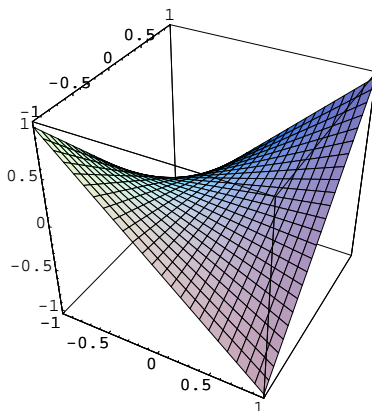
We integrate the second and third terms by parts, noticing that the boundary term vanishes at the endpoints because  $\delta(a) = \delta(b) = 0$ . This gives

$$\int_a^b \sum_{ij} \left[ \sum_k \frac{\partial g_{ij}}{\partial x_k} \delta_k \frac{d\gamma_i}{dt} \frac{d\gamma_j}{dt} - \frac{d}{dt} \left( g_{ij} \frac{d\gamma_j}{dt} \right) \delta_i - \frac{d}{dt} \left( g_{ij} \frac{d\gamma_i}{dt} \right) \delta_j \right] dt = 0$$

In the second term, change  $i$  to  $k$ ; in the third term, change  $j$  to  $k$ . This gives

$$\int_a^b \sum_k \left[ \sum_{ij} \frac{\partial g_{ij}}{\partial x_k} \frac{d\gamma_i}{dt} \frac{d\gamma_j}{dt} - \frac{d}{dt} \left( \sum_j g_{kj} \frac{d\gamma_j}{dt} \right) - \frac{d}{dt} \left( \sum_i g_{ik} \frac{d\gamma_i}{dt} \right) \right] \delta_k dt = 0$$

5. (12) Consider the saddle  $z = xy$ . Parameterize it in the standard way  $s(x, y) = (x, y, xy)$ .



- (a) If  $X = (1, 0)$  and  $Y = (0, 1)$ , determine  $\tilde{X}$  and  $\tilde{Y}$  at an arbitrary point  $(x, y, xy)$ .

**Answer:**  $\tilde{X} = (1, 0, y)$  and  $\tilde{Y} = (0, 1, x)$ .

- (b) It can be shown that at  $(1, 1, 1)$  we have  $g_{11} = 2, g_{12} = 1, g_{22} = 2$ . Compute  $\langle X, Y \rangle$ . Also compute  $\tilde{X} \cdot \tilde{Y}$ .

**Answer:**  $\langle X, Y \rangle = g_{11}(1)(0) + g_{12}((1)(1) + (0)(0)) + g_{22}(0)(1) = 1$  Also  $\tilde{X} = (1, 0, 1)$  and  $\tilde{Y} = (0, 1, 1)$  and their dot product is 1.

- (c) Notice that  $X = \frac{\partial}{\partial x}$  and let  $Y = \frac{\partial}{\partial y}$ . Compute  $X(Y)$  at the origin. Then compute  $b(X, Y)$  at the origin.

**Answer:** We must compute  $X(\tilde{Y}) = \frac{\partial}{\partial x}(0, 1, x) = (0, 0, 1)$ . We must write this answer in the form  $X(Y) = b(X, Y)n + \nabla_X Y$ . But in our case  $X(Y)$  is already normal to the surface and the magnitude of that normal is  $b(X, Y) = 1$ .

- (d) By a formula in class,  $b(X, Y) = -\langle BX, Y \rangle$ . Verify this at the origin by computing  $B(X) = X(n)$  there. You may use the fact that  $n = \frac{(-y, -x, 1)}{\sqrt{1+x^2+y^2}}$ .

**Answer:** At the origin, the  $g_{ij} = \delta_{ij}$ , so it suffices to show that the ordinary dot product of  $X(n)$  and  $Y$  is  $-1$ . But

$$X(n) = \frac{\partial}{\partial x} \frac{(-y, -x, 1)}{\sqrt{1+x^2+y^2}} = \frac{(0, -1, 0)}{\sqrt{1+x^2+y^2}} - \frac{1}{2}(1+x^2+y^2)^{-3/2}(2x)(-y, -x, 1).$$

When  $x = y = 0$ , this equals  $(0, -1, 0)$  or equivalently in coordinates  $(0, -1)$  and its dot product with  $Y$  is indeed  $-1$ .