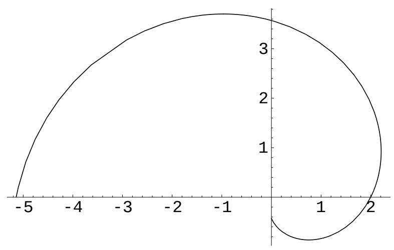
Mathematics 433/533 Midterm

April 29, 2005

Name ____

1. (12) The curve $\gamma(t) = ((t+2)\cos t, (t+2)\sin t)$ goes through the point $(2,0)$ when $t = 0$.



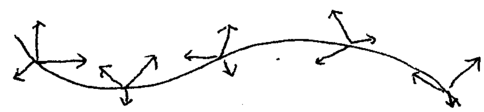
(a) Find the unit tangent vector T to the curve at $t = 0$.

(b) Find the unit normal N at $t = 0$. You can do this by looking at the above T and drawing a picture, without much calculation.

(c) Find the curvature κ at $t = 0$.

(d) Find the center of the osculating circle at $t = 0$.

2. (12) Let $\gamma(t)$ be a parameterized curve. Suppose $X_1(t), X_2(t)$, and $X_3(t)$ are three orthonormal vectors attached to $\gamma(t)$ for each t. See the picture below. These vectors need not be the vectors T, N, and B.

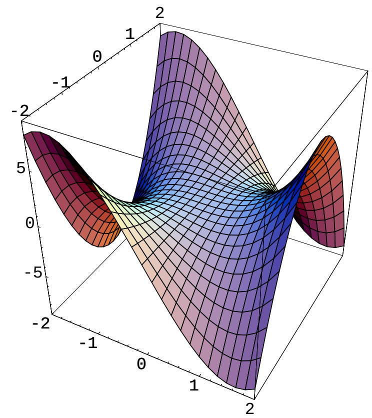


Write

$$\frac{dX_i}{dt} = \sum_j a_{ij} X_j$$

and let $A = (a_{ij})$ be the resulting matrix. Prove that $A^T = -A$.

3. (12) The surface $z = x^3 - 2xy^2$ is called a *monkey saddle.*



(a) Find a normal vector at $(1, 1, -1)$.

(b) Compute the g_{ij} at $(1, 1)$.

(c) Let $X = (1, 1)$ and $Y = (1, 2)$ be vectors at $(1, 1)$. Find the fancy length $||X||$ and the fancy inner product $< X, Y >$.

(d) Find the corresponding tangent vectors $\tilde{X}$ and $\tilde{Y}$ on the surface, and compute their dot product. You should get a number which you have already obtained. Why?

4. (12) In the derivation of the geodesic equation, we assumed that a curve $\gamma(t)$ for $a \leq t \leq b$ minimizes energy among all curves from $\gamma(a)$ to $\gamma(b)$. If $\delta(t)$ is a variation of the curve, we considered a family of new curves

$$\gamma_u(t) = \gamma(t) + u\delta(t)$$

The energy of these new curves is

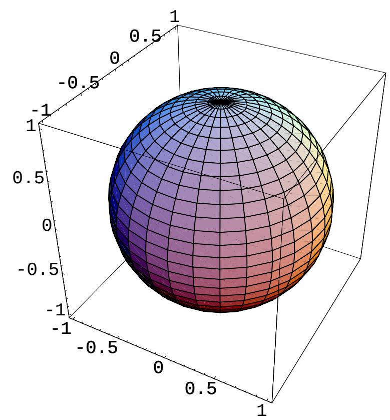
$$\mathcal{E}_u = \int_a^b \sum_{ij} g_{ij}(\gamma(t) + u\delta(t)) \frac{\partial}{\partial t}(\gamma_i(t) + u\delta_i(t)) \frac{\partial}{\partial t}(\gamma_j(t) + u\delta_j(t))\ dt.$$

Since $\mathcal{E}_u$ is minimal when $u = 0$, the derivative of $\mathcal{E}_u$ with respect to u must vanish at $u = 0$. Carry out the calculation of this derivative until you obtain an expression of the form below involving $\delta(t)$ but not its derivative. You need not simplify once you get this expression.

$$\int_a^b \sum_k (\text{complicated expression in } t)\, \delta_k(t)\ dt$$

When the calculation is done, you'll have an expression with an integral sign. End by explaining *in words* why the expression with an integral sign can be converted to a pure differential equation with no integral sign and no δ.

5. (12) Consider standard spherical coordinates $(\phi, \theta) \to (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$ on the sphere. Recall that ϕ is the angle down from the north pole and θ is the angle in the xy-plane. On maps, ϕ gives latitude and θ gives longitude.



(a) Compute g_{11}, g_{12}, g_{22} for this parameterization.

(b) Compute Γ^1_{22}. Please give details; don't just give the answer.

(c) As a check of your previous answer, $\Gamma^1_{22} = \sin\phi\cos\phi$ and $\Gamma^2_{12} = \frac{\cos\phi}{\sin\phi}$ and all other Christoffel symbols are zero. Using this fact, write the differential equations which geodesics must satisfy. You can answer this and remaining questions even if you made mistakes in a) and b).

(d) What do the differential equations say about geodesics with θ constant? Draw one of these geodesics on the above picture.

(e) What do the differential equations say about geodesics with ϕ constant? Explain this result in the picture.