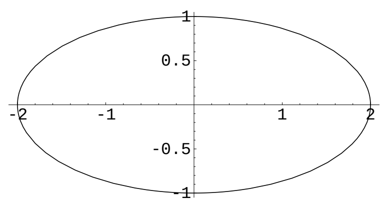
Mathematics 433/533 Midterm 2

May 13, 2005

Name ____

1. (12) Consider the ellipse $x^2+4y^2 = 4$. This ellipse can be parameterized by $\gamma(t) = (2\cos t, \sin t)$; this is *not* a parameterization by arc length.



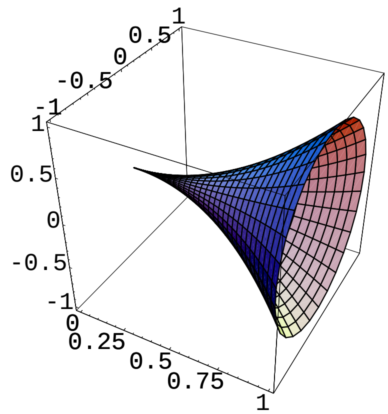
(a) Find the unit tangent vector T to the curve at $t = \frac{\pi}{4}$.

(b) Find the curvature κ at $t = 0$ when γ is at the point $(2, 0)$.

(c) Find the center of the osculating circle at $(2, 0)$.

2. (12) Consider the surface obtained by rotating the curve $y = x^2$ about the x-axis. This surface can be parameterized by $s(x, \theta) = (x, x^2 \cos\theta, x^2 \sin\theta)$ and then the unit normal is

$$n = \frac{(2x, -\cos\theta, -\sin\theta)}{\sqrt{1+4x^2}}.$$

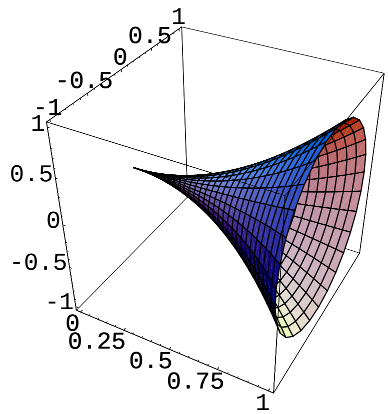


(a) Let $X = (1,0) = \frac{\partial}{\partial x}$ and $Y = (0,1) = \frac{\partial}{\partial \theta}$. Compute $Y(n)$. Show that your answer is a multiple of $\tilde{Y}$.

(b) It can be shown that $X(n) = \frac{2}{(1+4x^2)^{3/2}}\tilde{X}$. Using this result and your calculation from the previous question, write down the matrix for B.

(c) What are κ_1 and κ_2? Using just these formulas for κ_1 and κ_2, explain why the surface must look liike a saddle locally, as the picture suggests.

3. (12) Consider once more the surface parameterized by $s(x, \theta) = (x, x^2 \cos\theta, x^2 \sin\theta)$.



(a) Compute the g_{ij}.

(b) Compute Γ^1_{11}

(c) Compute the fancy length of the vector $(1, 1)$ at an arbitrary point (x, θ).

(d) Draw the curve $\gamma(t) = (t, t)$ for $0 \leq t \leq 2\pi$) on the picture of the surface above. Then write down an integral giving the length of this curve. Please base your integral on fancy inner products. You need not evaluate the integral.

4. (12) In the derivation of the geodesic equation, we assumed that a curve $\gamma(t)$ for $a \leq t \leq b$ minimizes energy among all curves from $\gamma(a)$ to $\gamma(b)$. If $\delta(t)$ is a variation of the curve, we considered a family of new curves

$$\gamma_u(t) = \gamma(t) + u\delta(t)$$

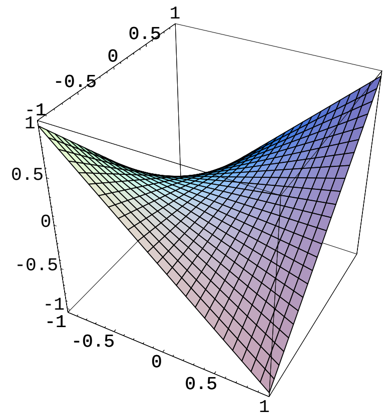
The energy of these new curves is

$$\mathcal{E}_u = \int_a^b \sum_{ij} g_{ij}(\gamma(t) + u\delta(t)) \frac{\partial}{\partial t}(\gamma_i(t) + u\delta_i(t)) \frac{\partial}{\partial t}(\gamma_j(t) + u\delta_j(t))\ dt.$$

Since $\mathcal{E}_u$ is minimal when $u = 0$, the derivative of $\mathcal{E}_u$ with respect to u must vanish at $u = 0$. Carry out the calculation of this derivative until you obtain an expression of the form below involving $\delta(t)$ but not its derivative. You need not simplify once you get this expression.

$$\int_a^b \sum_k (\text{complicated expression in } t)\, \delta_k(t)\ dt$$

5. (12) Consider the saddle $z = xy$. Parameterize it in the standard way $s(x, y) = (x, y, xy)$.



(a) If $X = (1, 0)$ and $Y = (0, 1)$, determine $\tilde{X}$ and $\tilde{Y}$ at an arbitrary point (x, y, xy).

(b) It can be shown that at $(1, 1, 1)$ we have $g_{11} = 2, g_{12} = 1, g_{22} = 2$. Compute $< X, Y >$. Also compute $\tilde{X} \cdot \tilde{Y}$.

(c) Notice that $X = \frac{\partial}{\partial x}$ and let $Y = \frac{\partial}{\partial y}$. Compute $X(Y)$ at the origin. Then compute $b(X, Y)$ at the origin.

(d) By a formula in class, $b(X, Y) = - < BX, Y >$. Verify this at the origin by computing $B(X) = X(n)$ there. You may use the fact that $n = \frac{(-y, -x, 1)}{\sqrt{1+x^2+y^2}}$.