Solutions #1, April 4, 2005

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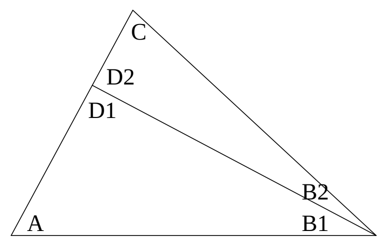
1: I'm going to assume that the sphere has radius one.

My first claim is that it suffices to prove the exercise for small triangles. If we have a large triangle, we can cut it into small triangles along great circles. Miraculously, the expression $\alpha + \beta + \gamma - \pi$ is additive; the value of this expression on the large triangle is the sum of the values of this expression on the smaller triangles. For example, in the picture below we have $D1 + D2 = \pi$ and so

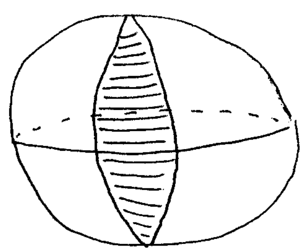
$$\Big(A + (B1 + B2) + C - \pi\Big) = (A + B1 + D1 - \pi) + (B2 + D2 + C - \pi)$$



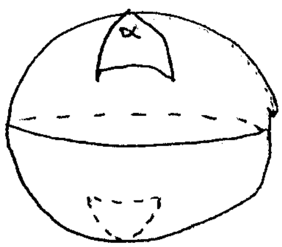
Since area is also additive, if $\alpha + \beta + \gamma - \pi$ equals area on the smaller triangles, it also equals area on the original large triangle.

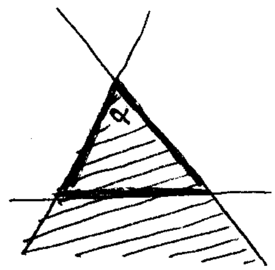
Suppose two great circles on the sphere meet, forming an angle α. Without loss of generality, we can suppose these circles meet at the north and south poles. The area inside this angle then forms a segment of the sphere which the book calls a "lune." I'll just call it a segment. The area of this segment equals

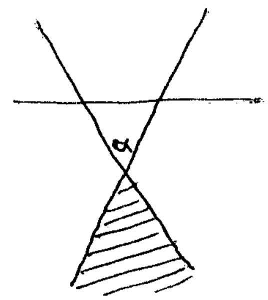
$$\text{area } = (\text{area of sphere}) \times (\text{fraction of a complete circle traced by } \alpha) = 4\pi \times \frac{\alpha}{2\pi} = 2\alpha$$



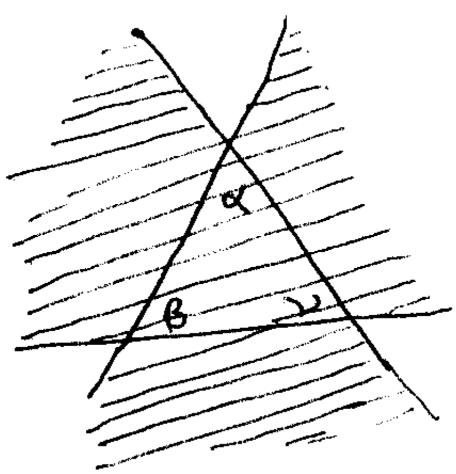
Now suppose the angle α at the north pole is one of the vertices of a triangle. Notice that there is a mirror image of this triangle at the south pole formed by the other intersections of the great circles which form the sides of the original triangle. The segment or lune traced in the previous paragraph covers the triangle at the north pole, but it meets the mirror image triangle “from the outside.” The last two pictures below show this segment as a shaded region near the north triangle, and as a shaded region near the south triangle.

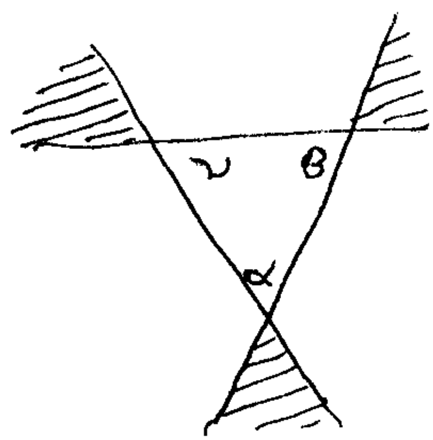






We can form additional segments or lunes starting at the other angles β and γ of the triangle near the north pole. These also cover the north triangle and reach the south triangle from the outside, as shown below.





Let us also form lunes starting at the three inside angles of the triangle at the south. These cover the south triangle, but reach the angles of the north triangle "from the outside." Altogether, we have six lunes which cover the entire sphere, but they cover the north triangle three times and the south triangle three times. Removing two of the three areas from the north and two of the three areas from the south gives

$$\text{area of sphere} = (\text{sum of areas of six lunes}) - 2(\text{area of triangle}) - 2(\text{area of triangle})$$

and so

$$4\pi = (2\alpha + 2\beta + 2\gamma + 2\alpha + 2\beta + 2\gamma) - 2(\text{area of triangle}) - -2(\text{area of triangle})$$

or

$$\text{area of triangle} = \alpha + \beta + \gamma - \pi.$$

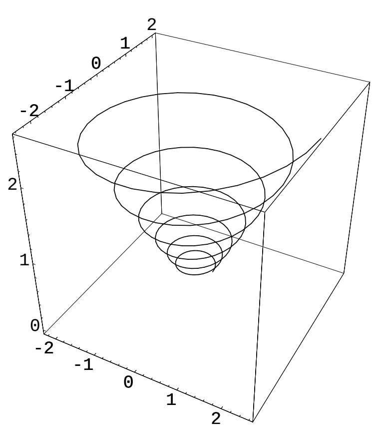
2: The length is

$$\int_1^2 \sqrt{(2)^2 + (2t)^2 + \left(\frac{1}{t}\right)^2}\, dt = \int_1^2 \sqrt{4t^2 + 4 + \frac{1}{t^2}}\, dt$$

And now the trick! By coincidence, the expression under the square root is $\left(2t + \frac{1}{t}\right)^2$, so the length is

$$\int_1^2 \left(2t + \frac{1}{t}\right)\, dt = t^2 + \ln t\Big|_1^2 = 3 + \ln 2$$

3: Below is a Mathematica picture. However I cheated and changed e^t to $e^{.05t}$ because the spiral in the original problem grows so rapidly with each twist that it is impossible to see what is happening.



4: I'll choose a starting point $t_0 = 0$. Then the length is of the curve is given by

$$u = \int_0^t \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2 + (e^t)^2}\, dt$$

The expression under the square root sign equals

$$e^{2t}\left(\cos^2 t - 2\cos t\sin t + \sin^2 t + \sin^2 t + 2\sin t\cos t + \cos^2 t + 1\right) = 3e^{2t}$$

so the length is given by

$$u = \int_0^t \sqrt{3}e^t\ dt = \sqrt{3}e^t\Big|_0^t = \sqrt{3}\left(e^t - 1\right)$$

This function is called $\varphi(t)$ in the lecture notes and the curve parameterized by arc length is given there as $\tau(u) = \gamma(\varphi^{-1}(u))$. In down-to-earth language, we should solve the equation $u = \sqrt{3}\left(e^t - 1\right)$ for t in terms of u, giving

$$t = \ln\left(\frac{u}{\sqrt{3}} + 1\right)$$

and then

$$\tau(u) = \gamma\left(\ln\left(\frac{u}{\sqrt{3}} + 1\right)\right) = \left(\left[\frac{u}{\sqrt{3}} + 1\right]\cos\left(\frac{u}{\sqrt{3}} + 1\right), \left[\frac{u}{\sqrt{3}} + 1\right]\sin\left(\frac{u}{\sqrt{3}} + 1\right), \left[\frac{u}{\sqrt{3}} + 1\right]\right)$$

5: We have $\gamma(t) = \tau(u(t))$. By the chain rule

$$\gamma'(t) = \tau'(u(t))\frac{du}{dt} = T(u(t))\frac{du}{dt}$$

$$\gamma''(t) = \left(\frac{dT(u(t))}{du}\frac{du}{dt}\right)\frac{du}{dt} + T(u(t))\frac{d^2u}{dt^2} = \kappa N\left(\frac{du}{dt}\right)^2 + T\frac{d^2u}{dt^2}$$

$$\frac{du}{dt} = ||\gamma'(t)|| \text{ by the fundamental theorem of calculus, since } u = \textstyle\int_0^t ||\gamma'||$$

$$\frac{d^2u}{dt^2} = \frac{d}{dt}\sqrt{\gamma'\cdot\gamma'} = \frac{1}{2\sqrt{\gamma'\cdot\gamma'}}\left(\gamma''\cdot\gamma' + \gamma'\cdot\gamma''\right) = \frac{\gamma'\cdot\gamma''}{\sqrt{\gamma'\cdot\gamma'}}$$

Solving the second of these equations for κN gives

$$\kappa N = \frac{\gamma'' - T\frac{d^2u}{dt^2}}{\left(\frac{du}{dt}\right)^2} = \frac{\gamma'' - \frac{\gamma'}{||\gamma'||}\frac{d^2u}{dt^2}}{\left(\frac{du}{dt}\right)^2} = \frac{\gamma'' - \frac{\gamma'(\gamma'\cdot\gamma'')}{||\gamma'||^2}}{||\gamma'||^2} = \frac{\gamma''||\gamma'||^2 - \gamma'\left(\gamma'\cdot\gamma''\right)}{||\gamma'||^4}$$

Hence

$$\kappa = \left|\left|\frac{\gamma''||\gamma'||^2 - \gamma'\left(\gamma'\cdot\gamma''\right)}{||\gamma'||^4}\right|\right|$$

Moreover, N is this vector divided by its length. In the next exercise set, we will discover that this formula for κ can be simplified. Perhaps you can see that already.

6: Parameterize the curve via $\gamma(t) = (t, t^2, 0)$. Then $\gamma' = (1, 2t, 0)$ and $\gamma'' = (0, 2, 0)$ and these expressions at $t = 1$ equal $\gamma' = (1, 2, 0)$ and $\gamma'' = (0, 2, 0)$. From exercise five we have

$$\kappa N = \frac{(0,2,0)\sqrt{5}^2 - (1,2,0)4}{\sqrt{5}^4} = \frac{(0,10,0)-(4,8,0)}{25} = \frac{(-4,2,0)}{25}$$

Therefore, κ is the length of this vector, $\frac{\sqrt{20}}{25}$, and N is this vector divided by its length, and so $N = \frac{(-4,2,0)}{\sqrt{20}}$. The radius of the osculating circle is $R = \frac{1}{\kappa} = \frac{25}{\sqrt{20}}$.

We can get to the center of the osculating circle by starting at the origin and moving to the point $\gamma(t_0)$ on the curve, and then moving in the direction N a distance R. Hence the center is

$$(1,1,0) + RN = (1,1,0) + \frac{25}{\sqrt{20}} \frac{(-4,2,0)}{\sqrt{20}}$$

Simplifying, the center equals

$$(1,1,0) + \left(-5, \frac{5}{2}, 0\right) = \left(-4, \frac{7}{2}, 0\right)$$

Here is a picture: