Solutions #2, April 8, 2005

1: The formula obtained in assignment 1 was

$$\kappa = \left|\left| \frac{\gamma''||\gamma'||^2 - \gamma'(\gamma' \cdot \gamma'')}{||\gamma'||^4} \right|\right|$$

Recall that $||v|| = \sqrt{v \cdot v}$. Hence the above expression equals

$$\kappa = \frac{\sqrt{(\gamma''||\gamma'||^2 - \gamma'(\gamma' \cdot \gamma'')) \cdot (\gamma''||\gamma'||^2 - \gamma'(\gamma' \cdot \gamma''))}}{||\gamma'||^4}$$

The expression under the square root can be expanded to be

$$(\gamma'' \cdot \gamma'')||\gamma'||^4 - (\gamma' \cdot \gamma'')^2||\gamma'||^2 - (\gamma' \cdot \gamma'')^2||\gamma'||^2 + (\gamma' \cdot \gamma'')^2||\gamma'||^2$$

which equals

$$||\gamma''||^2||\gamma'||^4 - (\gamma' \cdot \gamma'')^2||\gamma'||^2$$

Factoring $||\gamma'||^2$ out of this expression, and thus factoring $||\gamma'||$ out of the square root, gives

$$\kappa = \frac{\sqrt{||\gamma''||^2||\gamma'||^2 - (\gamma' \cdot \gamma'')^2}}{||\gamma'||^3}$$

The expression under the square root is now

$$(\text{the length of } \gamma'')^2(\text{the length of } \gamma')^2 - (\text{the length of } \gamma'')^2(\text{the length of } \gamma')^2 \cos^2\theta$$

where θ is the angle between γ' and γ''. Since $1 - \cos^2\theta = \sin^2\theta$, the formula for κ becomes

$$\kappa = \frac{||\gamma'||\ ||\gamma''||\ |\sin\theta|}{||\gamma'||^3}$$

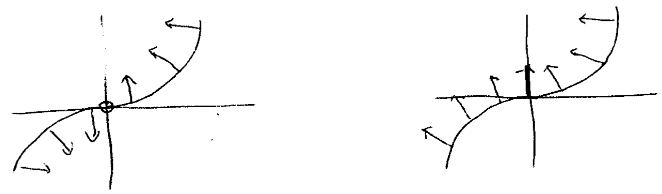
But recall that $||v \times w||$ is the length of v times the length of w times the sine of the angle between them (or equivalently, the area of the parallelogram spanned by v and w). So

$$\kappa = \frac{||\gamma' \times \gamma''||}{||\gamma'||^3}$$

2: The curve $y = f(x)$ can be parameterized as $\gamma(t) = (t, f(t), 0)$. Then $\gamma' \times \gamma''$ equals $(1, f', 0) \times (0, f'', 0) = (0, 0, f'')$ and so the formula in exercise 1 gives

$$\kappa = \frac{|f''|}{(1 + (f')^2)^{3/2}}$$

3 and 5: The geometric idea behind these problems is shown in the picture below, which projects the curves to the xy-plane. On the left we see the vectors N defined in the course. Notice the discontinuity at the origin. On the right we see the vectors after we changed the sign of N when $t < 0$. This makes it possible to extend N continuously to the origin.



Of course the actual curve is three dimensional, so we need a calculation to show that the picture is not misleading. An amazing thing happens in this calculation: the formulas for N and κ become simpler when we make the sign change. You'd think that we'd need separate formulas for $t < 0$ and $t > 0$, but notice that we don't.

Using earlier formulas from the course, the curvature is

$$\kappa = \frac{||(1, 3t^2, 1) \times (0, 6t, 0)||}{||(1, 3t^2, 1)||^{3/2}} = \frac{||(-6t, 0, 6t)||}{(2 + 9t^4)^{3/2}} = \frac{6|t|\sqrt{2}}{(2 + 9t^4)^{3/2}}$$

From exercise 5 on assignment 1,

$$\kappa N = \frac{\gamma''||\gamma'||^2 - \gamma'(\gamma' \cdot \gamma'')}{||\gamma'||^4} = \frac{(0, 6t, 0)(2 + 9t^4) - (1, 3t^2, 1)18t^3}{(2 + 9t^4)^2} = \frac{(-18t^3, 12t, -18t^3)}{(2 + 9t^4)^2}$$

Thus

$$N = \kappa N \; \frac{1}{\kappa} = \left(\frac{6t(-3t^2, 2, -3t^2)}{(2 + 9t^4)^2} \right) \left(\frac{(2 + 9t^4)^{3/2}}{6|t|\sqrt{2}} \right) = \frac{t}{|t|} \frac{(-3t^2, 2, -3t^2)}{\sqrt{4 + 18t^4}}$$

Notice that $\frac{(-3t^2,2,-3t^2)}{\sqrt{4+18t^4}}$ is a vector of length one. At the origin, it equals $(0, 1, 0)$. However, $\frac{t}{|t|}$ equals -1 for $t < 0$ and 1 for $t > 0$, and it is undefined for $t = 0$. So N is discontinuous at $t = 0$.

Since problem five is related to problem three, let's solve it next. There are two ways to solve this problem depending on whether we change the sign of N for negative t or change it for positive t. I'll change for negative t. Define a new $N(t)$ to equal the original one for $t > 0$ and the negative of the original one for $t < 0$. Thus

$$N(t) = \frac{(-3t^2, 2, -3t^2)}{\sqrt{2}\sqrt{2 + 9t^4}}$$

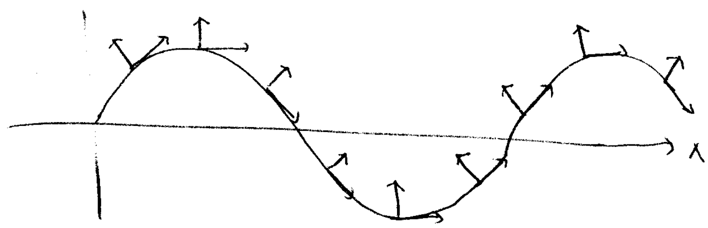
We must compensate by changing the sign of κ for $t < 0$. Thus we choose

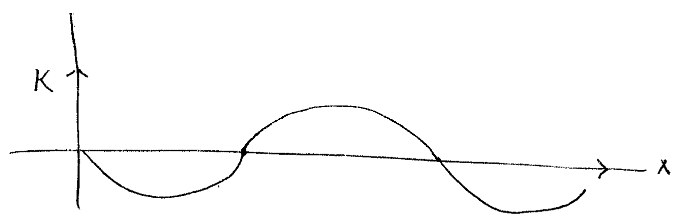
$$\kappa = \frac{6t\sqrt{2}}{(2 + 9t^4)^{3/2}}$$

Notice that neither expression is singular at $t = 0$, although $\kappa(0) = 0$.

As predicted, these formulas are simpler than the earlier formulas obtained at the start of the problem. We have eliminated absolute values and the annoying term $\frac{t}{|t|}$.

4: The geometric idea behind this problem is shown below. Notice that the normal vectors in the picture are the same as those in our course when the curve is below the x axis, but the opposite when the curve is above the axis. So the sign of κ has to change and become negative at the two ends of the graph.





Notice below that the formulas using the modified N and κ are simpler than the original formulas from the course.

Working first with unmodified formulas from the course, we have $\gamma = (t, \sin t, 0), \gamma' = (1, \cos t, 0)$, and $\gamma'' = (0, -\sin t, 0)$. Hence

$$\kappa = \frac{||(1, \cos t, 0) \times (0, -\sin t, 0)||}{(1 + \cos^2 t)^{3/2}} = \frac{(0, 0, -\sin t)}{(1 + \cos^2 t)^{3/2}} = \frac{|\sin t|}{(1 + \cos^2 t)^{3/2}}$$

Then

$$\kappa N = \frac{\gamma''||\gamma'||^2 - \gamma'(\gamma' \cdot \gamma'')}{||\gamma'||^4} = \frac{(0, -\sin t, 0)(1 + \cos^2 t) - (1, \cos t, 0)(-\sin t \cos t)}{(1 + \cos^2 t)^2}$$

and after simplifying,

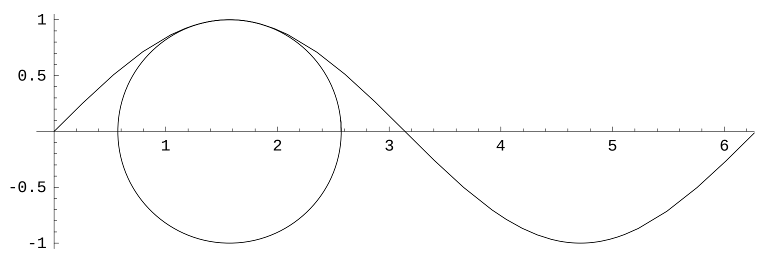
$$\kappa N = -\sin t \frac{(-\cos t, 1, 0)}{(1+\cos^2 t)^2}$$

Consequently

$$N = \kappa N \,\frac{1}{\kappa} = -\sin t \frac{(-\cos t, 1, 0)}{(1+\cos^2 t)^2} \,\frac{(1+\cos^2 t)^{3/2}}{|\sin t|} = -\frac{\sin t}{|\sin t|} \,\frac{(-\cos t, 1, 0)}{\sqrt{1+\cos^2 t}}$$

Notice that this N is a unit vector.

Let us stop and use these formulas to produce the osculating circle at $x = \frac{\pi}{2}$. We have $\kappa = 1$ and $N = (0, -1, 0)$ and thus the radius of the circle should be 1 and the center should be $\left(\frac{\pi}{2}, 0, 0\right)$, giving the following picture:



Now we must change the sign of N appropriately. As in the picture, we want to change the sign of N at exactly spots when $\sin t > 0$. But notice that $-\frac{\sin t}{|\sin t|}$ equals -1 when $\sin t > 0$ and equals 1 when $\sin t < 0$. So the easy way to change the sign of N appropriately is just to eliminate this entire term!

$$N = \frac{(-\cos t, 1, 0)}{\sqrt{1+\cos^2 t}}$$

A little thought shows that the appropriate formula for κ with similar sign changes is

$$\kappa = -\frac{\sin t}{(1+\cos^2 t)^{3/2}}$$

Please look back and notice that these formulas agree with the pictures which began our solution of this problem.

6: In each of the last three problems, we can translate and rotate our unknown solution until its initial conditions are conditions we prefer. Then we will solve the Frenet-Serret equations explicitly. The unknown solution is then obtained from our explicit solution by undoing the original rotation and translation.

In problem 6, rotate and translate the unknown solution until $\gamma(0) = (0,0,0), T(0) = (1,0,0), N(0) = (0,1,0)$, and $B(0) = (0,0,1)$. Since $\kappa = 0$, we have $\frac{d\gamma}{du} = T$, $\frac{dT}{du} = 0$, $\frac{dN}{du} = \tau B$, and $\frac{dB}{du} = -\tau N$. The two equations involving γ and T are independent of N and B, and the equations involving N and B are independent of γ and T, so it suffices to handle each pair of equations separately.

From the first pair it follows that $\frac{d^2\gamma}{du^2} = 0$ and so each coefficient of γ is linear:

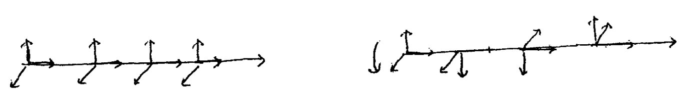
$$\gamma(u) = (a_1 u + b_1, a_2 u + b_2, a_3 u + b_3)\,.$$

Since $\gamma(0) = (0,0,0)$ and $\frac{d\gamma}{du}(0) = T(0) = (1,0,0)$, we conclude that $\gamma(u) = (u,0,0)$, which is a straight line.

If $\tau = 0$, the second pair of equations requires $\frac{dN}{du} = 0$ and $\frac{dB}{du} = 0$. So N and B are constant. The initial conditions then require that $N(u) = (0,1,0)$ and $B(u) = (0,0,1)$ for all u.

If $\tau \neq 0$, notice that $N(u) = \cos(\tau u)(0,1,0) + \sin(\tau u)(0,0,1)$ and $B(t) = -\sin(\tau u)(0,1,0) + \cos(\tau u)(0,0,1)$ solve the last pair of equations with the correct initial conditions. This solution then requires that the frame slowly rotate as we travel along the line.

Here are pictures of the solutions in these two cases:



7: This time we work backwards. If the curvature is κ, we expect the radius to be $\frac{1}{\kappa}$. Consequently, consider the curve

$$\gamma(u) = \left(\frac{1}{\kappa}\cos(\kappa u), \frac{1}{\kappa}\sin(\kappa u), 0\right)$$

Now *define the following.* We are not calculating these from γ, but rather just defining them out of thin air!

$$T(u) = (-\sin(\kappa u), \cos(\kappa u), 0)$$

$$N(u) = (-\cos(\kappa u), -\sin(\kappa u), 0)$$

$$B(u) = (0,0,1)$$

We easily check that these formulas satisfy the Frenet-Serret formulas for κ and $\tau = 0$, provided the initial conditions are

$$\gamma(0) = \left(\frac{1}{\kappa}, 0, 0\right)$$

$$T(0) = (0,1,0)$$

$$N(0) = (-1,0,0)$$

$$B(0) = (1,0,0)$$

Hence by uniqueness of solutions, the solution of the Frenet-Serret equations with constant $\kappa \neq 0$ and $\tau = 0$ and these initial conditions is a circle. However, the initial T, N, B listed here form a right handed coordinate system, so any other solution of the Frenet-Serret equations with constant $\kappa \neq 0$ and $\tau = 0$ can be rotated and translated so its initial conditions are these. Thus that solution, after rotation and translation, is a circle, and so that solution before rotation and translation is some other circle.

8: The argument is exactly the same, so we only get it started. Given κ and τ, we need to guess a solution of the Frenet-Serret equations which has the given κ and τ. For a moment, ignore the given κ and τ; instead let us compute the curvature and torsion of a helix.

We guess the solution is some helix $\gamma(t) = (a\cos t, a\sin t, bt)$. Earlier we parameterized this helix by arc length, so a better guess is

$$\gamma(u) = \left(a\cos\frac{u}{\sqrt{a^2+b^2}}, a\sin\frac{u}{\sqrt{a^2+b^2}}, \frac{bu}{\sqrt{a^2+b^2}}\right)$$

Then

$$T(u) = \frac{d\gamma}{du} = \left(-\frac{a}{\sqrt{a^2+b^2}}\sin\frac{u}{\sqrt{a^2+b^2}}, \frac{a}{\sqrt{a^2+b^2}}\cos\frac{u}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right)$$

and, since the curve is parameterized by arc length,

$$\frac{dT}{du} = \kappa N = \frac{a}{a^2+b^2}\left(-\cos\frac{u}{\sqrt{a^2+b^2}}, -\sin\frac{u}{\sqrt{a^2+b^2}}, 0\right)$$

Consequently, for the helix we have

$$\kappa = \frac{a}{a^2+b^2}$$

Then

$$B = T \times N = \left(-\frac{a}{\sqrt{a^2+b^2}}\sin\star, \frac{a}{\sqrt{a^2+b^2}}\cos\star, \frac{b}{\sqrt{a^2+b^2}}\right) \times (-\cos\star, -\sin\star, 0)$$

or

$$B = \left(\frac{b}{\sqrt{a^2+b^2}}\sin\star, -\frac{b}{\sqrt{a^2+b^2}}\cos\star, \frac{a}{\sqrt{a^2+b^2}}\right)$$

Recall that $\frac{dB}{du} = -\tau T$; in this particular case

$$\left(\frac{b}{a^2+b^2} cos\star, \frac{b}{a^2+b^2}\sin\star, 0\right) = -\tau\left(-\cos\star, -\sin\star, 0\right)$$

and so

$$\tau = \frac{b}{a^2+b^2}.$$

But remember that we are trying to guess a solution to the Frenet-Serret equations with given $\kappa \neq 0$ and $\tau \neq 0$. So we must choose a and b so $\kappa = \frac{a}{a^2+b^2}$ and $\tau = \frac{b}{a^2+b^2}$. A short calculation shows that the correct choices are

$$a = \frac{\kappa}{\kappa^2 + \tau^2}$$
$$b = \frac{\tau}{\kappa^2 + \tau^2}$$

Page 79, problem 6.6: We can parameterize this curve using θ instead of t:

$$\gamma(\theta) = (\rho(\theta)\cos\theta, \rho(\theta)\sin\theta)$$

Then the length is

$$\int_a^b \sqrt{(\rho'\cos\theta - \rho\sin\theta)^2 + (\rho'\sin\theta + \rho\cos\theta)^2}\; d\theta = \int_a^b \sqrt{(\rho')^2 + (\rho)^2}\; d\theta$$

The book asks for directed curvature. I'll compute κ up to absolute value out of laziness. First I compute $\gamma' \times \gamma''$:

$$(\rho'\cos\theta - \rho\sin\theta, \rho'\sin\theta + \rho\cos\theta, 0) \times (\rho''\cos\theta - 2\rho'\sin\theta - \rho\cos\theta, \rho''\sin\theta + 2\rho'\cos\theta - \rho\sin\theta, 0)$$

This equals

$$\left(0, 0, -\rho\rho'' + 2(\rho')^2 + \rho^2\right)$$

Consequently,

$$\kappa = \frac{||\gamma' \times \gamma''||}{||\gamma'||^{3/2}} = \frac{|-\rho\rho'' + 2(\rho')^2 + \rho^2|}{((\rho')^2 + \rho^2)^{3/2}}$$

Page 79, problem 6.8: We have

$$\int_{-\pi}^{\pi} \sqrt{(1+\cos t)^2 + \sin^2 t}\; dt = \int_{-\pi}^{\pi} \sqrt{2 + 2\cos t}\; dt = \sqrt{2}\int_{-\pi}^{\pi} \sqrt{1+\cos t}\; dt$$

Let's see if I can still integrate by hand. Substitute $t = 2u$, giving

$$\sqrt{2}\int_{-\pi/2}^{\pi/2} \sqrt{1+\cos 2u}\; 2du = \sqrt{2}\int_{-\pi/2}^{\pi/2} \sqrt{1 + \cos^2 u - \sin^2 u}\; 2du$$

This is

$$\sqrt{2}\int_{-\pi/2}^{\pi/2} \sqrt{\sin^2 u + \cos^2 u + \cos^2 u - \sin^2 u}\; 2du = 4\int_{-\pi/2}^{\pi/2} \cos u\; du = 4\sin u|_{-\pi/2}^{\pi/2} = 8$$

Of course we could use Mathematica, and it reports that the original integral is 8. Confirmation.

I got a secret pleasure out of doing this by hand. However, let me confess that I mistakenly made $dt = \frac{du}{2}$ until I checked with Mathematica and corrected the result. As Tom Lehrer says in a song, “well, that’s the wrong answer, but it’s the idea which counts.” Does anybody listen to Lehrer songs nowadays?

Page 91, problem 7.3: We have

$$T(s) = \left(\frac{4}{5} \sin s, -\cos s, \frac{3}{5} \sin s \right)$$

Notice that this vector has length one, so the curve is parameterized by arc length. Then

$$\frac{dT}{ds} = \kappa N = \left(\frac{4}{5} \cos s, \sin s, \frac{3}{5} \cos s \right)$$

and consequently $\kappa =$ and $N = \left(\frac{4}{5} \cos s, \sin s, \frac{3}{5} \cos s \right)$. Then

$$B = T \times N = \left(\frac{4}{5} \sin s, -\cos s, \frac{3}{5} \sin s \right) \times \left(\frac{4}{5} \cos s, \sin s, \frac{3}{5} \cos s \right) = \left(-\frac{3}{5}, 0, \frac{4}{5} \right)$$

Therefore $\frac{dB}{ds} = (0,0,0) = -\tau N$, so $\tau = 0$. This curve is a circle in the plane perpendicular to B.

Page 92, problem 7.5: Let us write this unknown ω as $\omega = aT + bN + cB$. Combining the desired formulas with the Frenet-Serret formulas, we have

$$T' = \kappa N = (aT + bN + cB) \times T = -bB + cN$$

$$N' = -\kappa T + \tau B = (aT + bN + cB) \times N = aB - cT$$

$$B' = -\tau N = (aT + bN + cB) \times B = -aN + bT$$

The first equation gives $b = 0$ and $c = \kappa$. The second equation gives $c = \kappa$ and $a = \tau$. The final equation gives $a = \tau$ and $b = 0$. Consequently

$$\omega = \tau T + \kappa B$$

Page 92, page 7.6: Since $\frac{dB}{du} = -\tau N$, the length of $\frac{dB}{du}$ is the absolute value of τ, which therefore is determined by B. Moreover,

$$\pm N = \frac{B'}{||B'||}.$$

Finally since $T = N \times B$ we have

$$\pm T = \frac{B'}{||B'||} \times B.$$

Interprete these two formulas in the following way: if the correct choice is the plus sign in the first case, it is also the plus sign in the second case. If it is the minus sign in the first case, it is the minus sign in the second case. So we have our hands on (T, N, B) or $(-T, -N, B)$ but we don't know which one.

Since $\frac{dT}{du} = \kappa N$ and $\frac{d(-T)}{du} = \kappa(-N)$, we can compute κ without knowing which case we are in.

Page 92, page 7.7: This exercise is another way to see that κ and τ completely determine the curve. This time, we show that if we know the power series expanion of κ and τ, we can determine the power series expansion of γ.

To simplify notation, I'll take $u_0 = 0$. Then by Taylor's theorem we have

$$\gamma(u) = \gamma(0) + \gamma'(0)u + \gamma''(0)\frac{u^2}{2!} + \gamma'''(0)\frac{u^3}{3!} + \dots$$

But $\gamma' = T$, so $\gamma'' = \frac{dT}{du} = \kappa N$, so $\gamma''' = \frac{d}{du}(\kappa N) = \kappa' N + \kappa \frac{dN}{du} = \kappa' N + \kappa(-\kappa T + \tau B)$, etc. Consequently, we can write down the complete power series, and its first few terms are

$$\gamma(u) = \gamma(0) + T(0)u + \kappa(0)N(0)\frac{u^2}{2!} + \left(\kappa'(0)N(0) - \kappa^2(0)T(0) + \kappa(0)\tau(0)B(0)\right)\frac{u^3}{3!} + \dots$$

This is more illuminating if we just write T, N, B for $T(0), N(0), B(0)$ and just write κ, κ', etc., for terms evaluated at zero. We have

$$\gamma(u) = \gamma(0) \quad + \quad \left(u - \frac{\kappa^2}{6}u^3 + \dots\right)T \quad + \quad \left(\frac{\kappa}{2}u^2 + \frac{\kappa'}{6}u^3 + \dots\right)N \quad + \quad \left(\frac{\kappa\tau}{6}u^3 + \dots\right)B$$

Notice that the initial conditions are clearly visible in the final expression, since we can choose the initial position $\gamma(0)$ and the initial frame T, N, B arbitrarily, and then all coefficient power series are completely determined by the power series expansions of κ and τ.