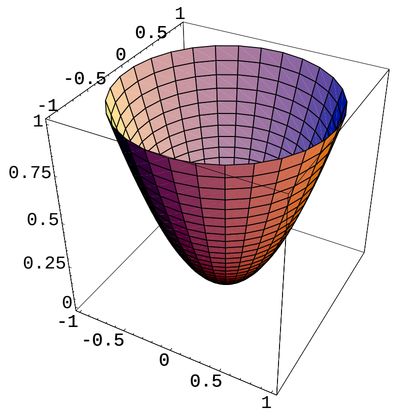
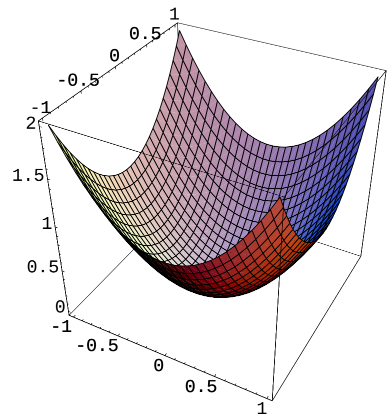
Solutions #3, April 15, 2005

1: Two pictures of the surface are shown below.





These pictures were made with *Mathematica* and it is interesting that the *Mathematica* commands to produce them use the terminology of our course:

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ParametricPlot3D[{r Cos[theta], r Sin[theta], r^2}, {r, 0, 1}, {theta, 0, 2 Pi},
          BoxRatios -> {1, 1, 1}]

ParametricPlot3D[{x, y, x^2 + y^2}, {x, -1, 1}, {y, -1, 1}]
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This example illustrates why one should use a variety of coordinate parameterizations; cylindrical coordinates bring out the symmetry of the surface which is somewhat hidden with rectangular coordinates.

We have $s(u,v) = (u, v, u^2 + v^2)$ so $\frac{\partial s}{\partial u} = (1, 0, 2u)$; at $(1, 2, 5)$ this derivative is $(1, 0, 2)$. Similarly $\frac{\partial s}{\partial v} = (1, 0, 2v)$; at $(1, 2, 5)$ this is $(0, 1, 4)$. So $n = (1, 0, 2) \times (0, 1, 4) = (-2, -4, 1)$. The tangent plane is then

$$\Big\{\ (x, y, z) \mid \Big((x, y, z) - (1, 2, 5)\Big) \cdot (-2, -4, 1) = 0\ \Big\}$$

and thus the equation of the plane is $z = 5 + 2(x - 1) + 4(y - 2)$, that is, $z = 2x + 4y - 5$.

2: Well, $27 = 2(2) + 4(7) - 5$.

We want to write $(1, 5, 22) = a(1, 0, 2) + b(0, 1, 4) = (a, b, 2a + 4b)$. Of course $a = 1$ and $b = 5$. So the corresponding coordinate vector is $(1, 5)$.

3: If $s(u,v) = (u,v,f(u,v))$, then $\frac{\partial s}{\partial u} = \left(1, 0, \frac{\partial f}{\partial u}\right)$ and $\frac{\partial s}{\partial v} = \left(0, 1, \frac{\partial f}{\partial v}\right)$. So

$$n = \left(1, 0, \frac{\partial f}{\partial u}\right) \times \left(0, 1, \frac{\partial f}{\partial v}\right) = \left(-\frac{\partial f}{\partial u}, -\frac{\partial f}{\partial v}, 1\right)$$

A tangent vector is a vector $v = (a,b,c)$ such that $v \cdot n = 0$. So $-a\frac{\partial f}{\partial u} - b\frac{\partial f}{\partial v} + c = 0$ and thus

$$v = \left(a, b, a\frac{\partial f}{\partial u} + b\frac{\partial f}{\partial v}\right)$$

To find the corresponding coordinate vector, we must write this as a linear combination of $\frac{\partial s}{\partial u}$ and $\frac{\partial s}{\partial v}$. The coefficients of this linear combination then give the coefficients of the coordinate vector. But

$$\left(a, b, a\frac{\partial f}{\partial u} + b\frac{\partial f}{\partial v}\right) = a\left(1, 0, \frac{\partial f}{\partial u}\right) + b\left(0, 1, \frac{\partial f}{\partial v}\right) = a\frac{\partial s}{\partial u} + b\frac{\partial s}{\partial v}$$

and thus the corresponding coordinate vector is (a,b).

4: We have $\frac{\partial s}{\partial u} = (2u, 1, v)$; at the point $u = v = 1$ this is $(2,1,1)$. Similarly $\frac{\partial s}{\partial v} = (0,1,u)$ and at the given point this is $(0,1,1)$. So $n = (2,1,1) \times (0,1,1) = (0,-2,2)$. The tangent plane is

$$\left\{ (x,y,z) \mid \Big((x,y,z) - (1,2,1)\Big) \cdot (0,-2,2) = 0 \right\}$$

and the equation of this plane is $-2(y-2) + 2(z-1) = 0$ and so $z = y - 1$. The point $(3,5,4)$ is on this plane because $4 = 5 - 1$; the point $(1,2,1)$ is on the plane because $1 = 2 - 1$. So the difference, $(2,3,3)$ is in the tangent space T.

We must write $(2,3,3)$ as a linear combination of $(2,1,1)$ and $(0,1,1)$. Writing

$$(2,3,3) = a(2,1,1) + b(0,1,1) = (2a, a+b, a+b)$$

we discover that $2 = 2a$ and $3 = a + b$, so $a = 1$ and $b = 2$. Thus the corresponding coordinate vector is $(1,2)$.

5: Recall that for surfaces of the form $z = f(x,y)$ we have $g_{11} = 1 + \left(\frac{\partial f}{\partial x}\right)^2$ and $g_{12} = g_{21} = \frac{\partial f}{\partial x}\frac{\partial f}{\partial y}$ and $g_{22} = 1 + \left(\frac{\partial f}{\partial y}\right)^2$. Hence for $z = x^2 + y^2$ we have $g_{11} = 1 + 4x^2$, $g_{12} = g_{21} = 4xy$, and $g_{22} = 1 + 4y^2$.

6: At the point $(1,2)$ we have $g_{11} = 5, g_{12} = g_{21} = 8$, and $g_{22} = 17$. The length of $X = (1,5)$ is the square root of $< X, X >= g_{11}X_1^2 + 2g_{12}X_1X_2 + g_{22}X_2^2 = 5(1)^2 + 2 \cdot 8(1)(5) + 17(5)^2 = 510$, and so $\sqrt{510} = 22.5832$.

By exercise two, the corresponding tangent vector in three space is $(1,5,22)$ and its length is $\sqrt{1^2 + 5^2 + 22^2} = \sqrt{510} = 22.5832$.

7: We have $< X, Y >= ||X||\ ||Y||\ \cos\theta$.

By exercise five, at $(1,1)$ we have $g_{11} = 5, g_{12} = g_{21} = 4, g_{22} = 5$. Hence for $X = (1,5)$ we have $||X||^2 = 5(1)^2 + 2\cdot 4(1)(5) + 5(5)^2 = 170$, so $||X|| = \sqrt{170}$. For $Y = (2,2)$ we have $||Y||^2 = 5(2)^2 + 2\cdot 4(2)(2) + 5(2)^2 = 72$, so $||Y|| = \sqrt{72}$. Finally $< X, Y >= 5(1)(2) + 4(1)(2) + 4(5)(2) + 5(5)(2) = 108$.

Consequently the first line above reads

$$108 = \sqrt{170}\sqrt{72}\cos\theta$$

and so

$$\cos\theta = \frac{108}{\sqrt{170}\sqrt{72}} = .976187$$

which gives

$$\theta = .218669 \text{ radians} = 12.53 \text{ degrees}$$

We can do this problem another way. The tangent vector corresponding to (a,b) is

$$\left(a, b, a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y}\right) = (a, b, a\cdot 2 + b\cdot 2)\,.$$

So the vector corresponding to $(1,5)$ is $(1,5,12)$ and the vector corresponding to $(2,2)$ is $(2,2,8)$. The length of the first of these vectors is $\sqrt{1^2 + 5^2 + 12^2} = \sqrt{170}$ and the length of the second vector is $\sqrt{2^2 + 2^2 + 8^2} = \sqrt{72}$ and their dot product is $1\cdot 2 + 5\cdot 2 + 12\cdot 8 = 108$. From here on the argument is as before because we got the same lengths and inner product.

8: The length of a curve is given by

$$\int_a^b ||\frac{d\gamma}{dt}||\ dt = \int_a^b \sqrt{< \frac{d\gamma}{dt}, \frac{d\gamma}{dt} >}\ dt = \int_a^b \sqrt{g_{11}\left(\frac{du}{dt}\right)^2 + 2g_{12}\frac{du}{dt}\frac{dv}{dt} + g_{22}\left(\frac{dv}{dt}\right)^2}\ dt$$

In our case the curve can be parameterized by θ as $(\cos\theta, \sin\theta)$ and expression under the square root sign becomes

$$(1 + 4x^2)(-\sin\theta)^2 + 2\cdot 4xy(-\sin\theta)(\cos\theta) + (1 + 4y^2)(\cos\theta)^2$$

However, this expression isn't legal because we are supposed to substitute $\cos\theta$ for x and $\sin\theta$ for y. The correct expression is

$$(1 + 4\cos^2\theta)(-\sin\theta)^2 + 2\cdot 4(\cos\theta)(\sin\theta)(-\sin\theta)(\cos\theta) + (1 + 4\sin^2\theta)(\cos\theta)^2$$

This expression then equals

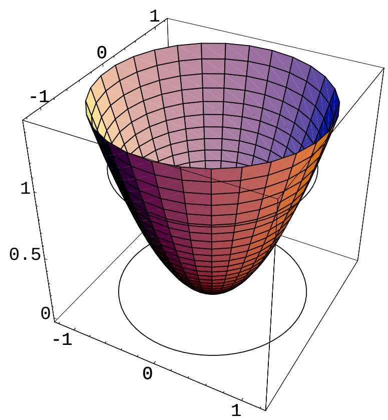
$$\sin^2\theta + 4\sin^2\theta\cos^2\theta - 8\sin^2\theta\cos^2\theta + \cos^2\theta + 4\sin^2\theta\cos^2\theta = 1$$

and so the length of the circle is just

$$\int_0^{2\pi} \sqrt{1}\ d\theta = 2\pi$$

as expected.

The reason we obtain this result is that the circle on the surface is exactly the circle in the coordinate plane lifted up one unit and so no distortion in length occurs:



9: Let us choose the curve $\gamma(t) = (t, 0)$ for $0 \leq t \leq 1$. The length of this curve is then

$$\int_0^1 \sqrt{(1+4x^2)(1)^2 + 2 \cdot 4xy(1)(0) + (1+4y^2)(0)^2}\ dt$$

except that we must substitute t for x and 0 for y, so

$$\int_0^1 \sqrt{(1+4t^2)}\ dt = 1.47894$$

using *Mathematica*. So the length of the circle is not $\pi(\text{radius})^2$ but rather smaller.

10: We have $g_{11} = 1 + \left(\frac{\partial f}{\partial x}\right)^2 = 1 + 4x^2$, $g_{12} = g_{21} = \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} = -4xy$ and $g_{22} = 1 + \left(\frac{\partial f}{\partial y}\right)^2 = 1 + 4y^2$. So the length is

$$\int_0^{2\pi} \sqrt{(1+4x^2)(-\sin\theta)^2 - 2 \cdot 4xy(-\sin\theta)(\cos\theta) + (1+4y^2)(\cos\theta)^2}\ d\theta$$

except that we must substitute $x = \cos\theta$ and $y = \sin\theta$, giving

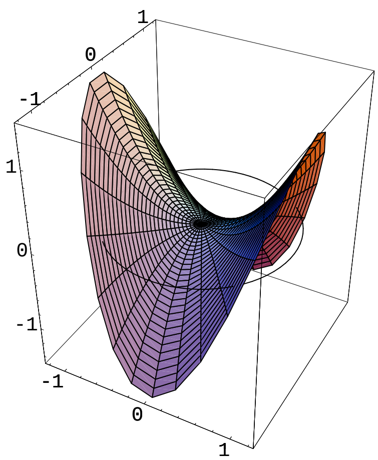
$$\int_0^{2\pi} \sqrt{(1+4\cos^2\theta)(-\sin\theta)^2 - 2 \cdot 4\cos\theta\sin\theta(-\sin\theta)(\cos\theta) + (1+4\sin^2\theta)(\cos\theta)^2}\ d\theta$$

This equals

$$\int_0^{2\pi} \sqrt{1 + 16\sin^2\theta\cos^2\theta}\ d\theta = 10.5407$$

rather than $2\pi r = 2\pi = 6.28$.

The length is longer because when we lift the circle up to the surface, it has to rise and fall as it goes around the saddle:



11: The line (t, t) becomes the line $(t, t, 0)$ on the saddle, so it has the expected length 1. But the line $(t, 0)$ becomes the curve $(t, 0, t^2)$ which was computed in problem nine to have length 1.4789. Hence these two "radii" have different lengths and the curve in question is not a circle on the saddle.