Solutions #4, April 22, 2005

1: We can parameterize by $s(x,y) = (x,y,xy)$ and then $\frac{\partial s}{\partial x} = (1,0,y)$ and $\frac{\partial s}{\partial y} = (0,1,x)$. So $g_{11} = \frac{\partial s}{\partial x} \cdot \frac{\partial s}{\partial x} = 1 + y^2$ and $g_{12} = \frac{\partial s}{\partial x} \cdot \frac{\partial s}{\partial y} = xy$ and $g_{22} = \frac{\partial s}{\partial y} \cdot \frac{\partial s}{\partial y} = 1 + x^2$. Of course we also have formulas for the g_{ij} directly in terms of $f(x,y)$, which can be used instead to get these results.

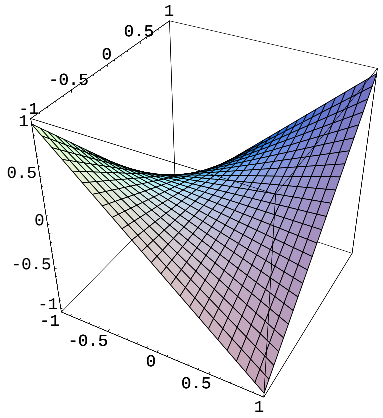
If $\gamma(t) = (t,0)$, then $\gamma'(t) = (1,0)$ and $< \gamma', \gamma' >= g_{11}1^2 + 2g_{12}(1)(0) + g_{22}0^2 = g_{11}(\gamma(t)) = 1 + 0^2 = 1$. Hence $\int_0^2 \sqrt{< \gamma', \gamma'}\, dt = 2$. This is just the length of the line in the coordinate plane.

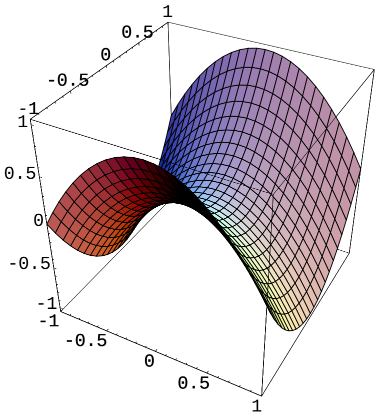
The reason is that this particular line in the plane is actually *on the saddle* without lifting or anything. Indeed, when $(x,y) = (t,0)$, we have $xy = (t)(0) = 0$, so $s(t,0) = (t,0,0)$.

You can visualize this line in the picture on the left below by intrsecting this surface with the xy-plane. Notice that the origin is at the center of the cube. A little thought shows that the xy-plane intersects the circle in two perpendicular lines, namely the x and y axes. The picture on the right is rotated 45 degrees, and these lines are the lines $y = \pm x$ on that xy-plane.

2: The derivative of this curve is $(0,-1)$ and so $||\gamma'||$ is the square root of $g_{11}0^2 + 2g_{12}(0)(-1) + g_{22}(-1)^2 = g_{22} = 1 + x^2$. We must evaluate this at $\gamma(t) = (1, 1-t)$, and the answer is $1 + 1^2 = 2$. So the length is $\int_{-1}^{1} \sqrt{2}\, dt = 2\sqrt{2}$. This is not the length of the coordinate line because it is only 2. But it is a rather simple answer.

The corresponding curve on the surface is $s(1, 1-t) = (1, 1-t, 1(1-t)) = (1, 1-t, 1-t)$. Notice that this is again a straight line. This particular line is immediately apparent in the picture on the left; it is the diagonal line running up the right side of the coordinate cube. This line forms the hypotenuse of a right triangle with sides 2 and 2, so its length is $\sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$.





3: When $\gamma(t) = (t, 1-t)$, the derivative is $(1,-1)$ and $< \gamma', \gamma' >$ equals $g_{11}1^2 + 2g_{12}(1)(-1) +$

$g_{22}(-1)^2 = g_{11} - 2g_{12} + g_{22}$. Inserting values for the g_{ij} calculated in problem 1, this is $(1+y^2) - 2xy + (1+x^2)$. We must evaluate at $(x, y) = (t, 1-t)$, giving $1 + (1-t)^2 - 2t(1-t) + 1 + t^2 = 3 - 4t + 4t^2$. Thus the length is

$$\int_{-1}^{1} \sqrt{3 - 4t + 4t^2}\, dt = \frac{1}{4}\left(\sqrt{3} + 3\sqrt{11} + 2\text{arcsinh}\frac{1}{\sqrt{2}} + 2\text{arcsinh}\frac{3}{\sqrt{2}}\right) = 3.99806$$

according to *Mathematica*.

The corresponding curve on the surface is $s(t, 1-t) = (t, 1-t, t(1-t)) = (t, 1-t, t-t^2)$ and this is no longer a line, hence the complicated answer. Indeed, it is a downward pointing parabola.

4: The length of e_1 is the square root of $g_{11}1^2 + 2g_{12}(1)(0) + g_{22}0^2 = g_{11} = 1 + y^2$, and thus $\sqrt{1+y^2}$. Similarly the length of e_2 is $\sqrt{1+x^2}$. The inner product of these two vectors is $< (1,0), (0,1) >= g_{11}(1)(0) + g_{12}\left((1)(1) + (0)(0)\right) + g_{22}(0)(1) = g_{12} = xy$ and so the cosine of the angle between these vectors is

$$\frac{< e_1, e_2 >}{||e_1||\ ||e_2||} = \frac{xy}{\sqrt{1+y^2}\sqrt{1+x^2}}$$

Hence

$$\theta = \arccos\left(\frac{xy}{\sqrt{1+y^2}\sqrt{1+x^2}}\right)$$

In the special case $(x, y) = (1, 1)$, this number is $\theta = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$, or 60 degrees. When $(x, y) = (1, -1)$ the number is $\theta = \arccos\left(\frac{-1}{2}\right) = \frac{2\pi}{3}$, which is 120 degrees.

To see that this is reasonable, look at the picture on the left on page one. The point $(1, 1)$ becomes the point $(1, 1, 1)$ on the surface, which is at the top right. The vectors e_1 and e_2 point along the sides of the cube, say on the bottom of the cube, and both point toward their meeting point. When we lift them to the surface, we get the sharp corner at upper right, and the vectors along the sides of this corner meet at a smaller angle. Indeed 60 degrees is a possibility!

The angle at $(1, -1)$ corresponds to the corner at the lower right. But this time one arrow points into the angle and the other points out of the angle. By symmetry the angle on the surface is the same 60 degrees, but now a complementary 120 degree angle is being determined by the arrows because only one points in to the intersection point.

5: If $y = x$, the angle is

$$\arccos\left(\frac{x^2}{\sqrt{1+x^2}\sqrt{1+x^2}}\right) = \arccos\left(\frac{x^2}{1+x^2}\right)$$

As x moves out from zero to infinity, this fraction moves from 0 to 1, so the arc cosine of the angles moves from $\arccos 0 = \frac{\pi}{2} = 90$ degrees to $\arccos 1 = 0$ degrees.

If $y = -x$we have

$$\theta = \arccos\left(\frac{-x^2}{\sqrt{1+x^2}\sqrt{1+x^2}}\right) = \arccos\left(\frac{-x^2}{1+x^2}\right)$$

. As x moves out from zero to infinity, this fraction moves from 0 to -1, and the arc cosine moves from 90 degrees to 180 degrees.

6: The length of e_1 is $\sqrt{1+y^2}$, and we should set X to e_1 divided by its length, so

$$X_1 = \left(\frac{1}{\sqrt{1+y^2}}, 0\right)$$

We want to chooe $X_2 = (a, b)$ perpendicular to X and thus perpendicular to e_1. So

$$< (a,b),(1,0) >= g_{11}(a)(1) + g_{12}((a)(0) + (b)(1)) + g_{22}(b)(0) = g_{11}a + g_{12}b$$

We want this inner product to be zero, so

$$a = -\frac{g_{12}}{g_{11}}b = -\frac{xy}{\sqrt{1+y^2}}b$$

Hence

$$Y = \left(-\frac{xy}{\sqrt{1+y^2}}, 1\right) b$$

where b must be chosen to make this vector have length one. Ignoring b, the length of the vector is

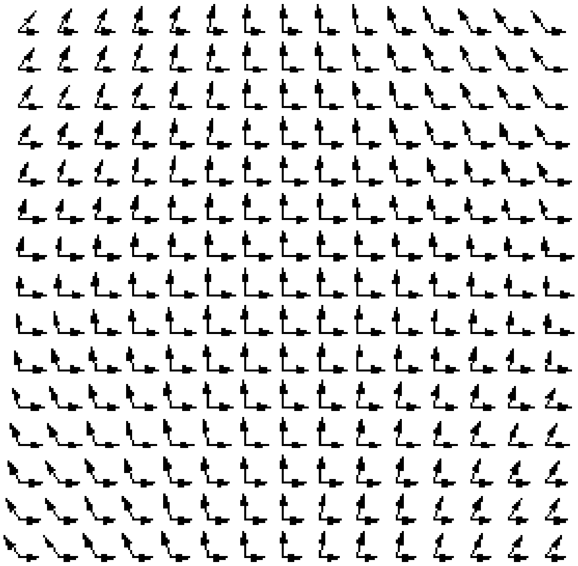
$$\left\{ g_{11}\left(\frac{-xy}{\sqrt{1+y^2}}\right)^2 + 2g_{12}\left(\frac{-xy}{\sqrt{1+y^2}}\right) + g_{22} \right\}$$

which equals

$$(1+y^2)\frac{x^2y^2}{1+y^2} + 2xy\frac{-xy}{\sqrt{1+y^2}} + 1 + x^2 = x^2y^2 - 2\frac{x^2y^2}{\sqrt{1+y^2}} + 1 + x^2$$

and thus Y is the vector $\left(-\frac{xy}{\sqrt{1+y^2}}, 1\right)$ divided by the square root of this expression.

Here is a picture of the resulting vector field.



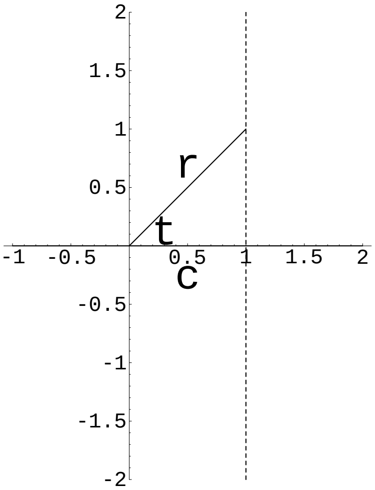
7: This is slightly tricky when the saddle is given by the equation $z = x^2 - y^2$, but it is certainly easy when the saddle is given by the equation $z = xy$. After reading the solution, look at the picture on page one again. The two lines are just $x =$ constant and $y =$ constant. Indeed, if we have $\gamma(t) = (c, t)$ in the coordinate plane, then on the saddle we have $\gamma(t) = (c, t, ct)$ and this is a line. The argument for $y =$ constant lines is the same.

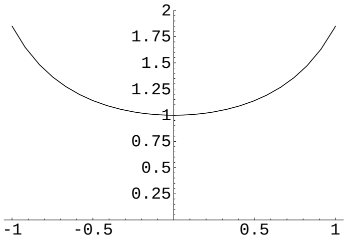
8: We have $\frac{\partial s}{\partial r} = (\cos\theta, \sin\theta, 0)$ and $\frac{\partial s}{\partial \theta} = (-r\sin\theta, r\cos\theta, 0)$. So $g_{11} = \frac{\partial s}{\partial r} \cdot \frac{\partial s}{\partial r} = 1$, $g_{12} = 0$, and $g_{22} = r^2$. Thus we obtain the familiar formula $ds = \sqrt{(dr)^2 + r^2(d\theta)^2}$ for distance using polar coordinates.

The picture below shows one particular line in the plane. In the picture, the angle θ is labeled t. From trigonometry, $\cos\theta = \frac{c}{r}$ and consequently $r = \frac{c}{\cos\theta}$. Consequently,

$$r = \frac{c}{\cos\theta}$$

is the formula for a line in polar coordinates. The picture on the right shows this curve plotted in the $r - \theta$ plane.

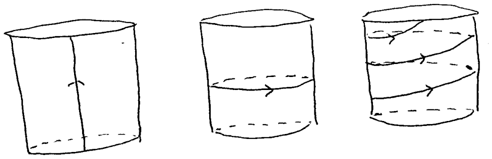




To get any other line (except radial lines where r cannot be written as a function of θ), just rotate. Rotation is easy — change θ to $\theta - \theta_0$, so $r = \frac{c}{\cos(\theta-\theta_0)}$. This rotates the line by θ_0. Do you see why the sign of θ_0 must be negative in the formula?

9: We have $\frac{\partial s}{\partial\theta} = (-\sin\theta, \cos\theta, 0)$ and $\frac{\partial s}{\partial h} = (0, 0, 1)$, so $g_{11} = (-\sin\theta, \cos\theta, 0)\cdot(-\sin\theta, \cos\theta, 0) = 1$, $g_{12} = 0$, and $g_{22} = 1$. Consequently, the metric tensor in coordinate space describes Euclidean distance, so geodesics in coordinates are straight lines.

When we wrap the coordinate plane around the cylinder, these straight lines becomes circles around the cylinder, vertical lines, or helixes. So these three types of curves form the geodesics on a cylinder.

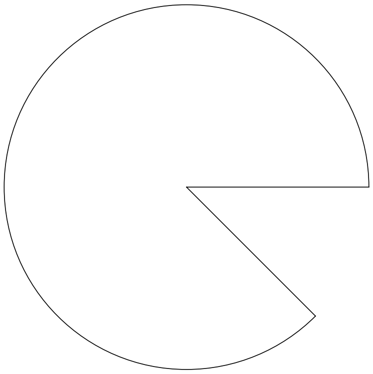


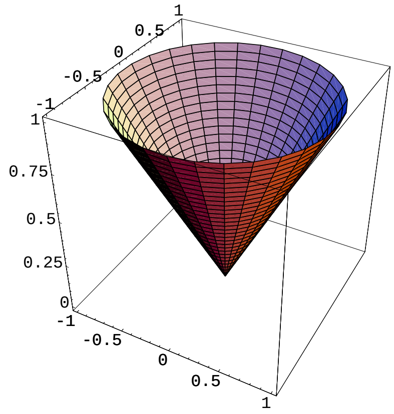
10: The two partial derivatives are $(\cos\theta, \sin\theta, 1)$ and $(-r\sin\theta, r\cos\theta, 0)$. Taking dot produces gives $g_{11} = 2, g_{12} = 0, g_{22} = r^2$. Consequently, the metric tensor is close to the polar metric. We can make it exactly the polar metric by producing new coordinates $\tilde{r}, \tilde{\theta}$ on the $r - \theta$ plane where $r = a\tilde{r}$ and $\theta = b\tilde{\theta}$. Then $2dr^2 + r^2 d\theta^2 = 2a^2 d\tilde{r}^2 + a^2\tilde{r}^2 b^2 d\tilde{\theta}^2$. If we choose a so $2a^2 = 1$ and b so

$a^2b^2 = 1$, then this is $d\tilde{r}^2 + \tilde{r}^2 d\tilde{\theta}^2$ and so exactly polar coordinates in terms of $\tilde{r}$ and $\tilde{\theta}$. Notice that the proper choice of a and b is $a = \frac{1}{\sqrt{2}}$ and $b = \sqrt{2}$. The $r - \theta$ plane is obtained from the $\tilde{r} - \tilde{\theta}$ plane by scaling the horizontal and vertical distances using different scales, so the geodesics are still the curves shown in the right hand picture of the solution to problem 8, but just rescaled.

11: We'll actually redo the calculation in the previous exercise, but from a different point of view. We are going to fold the region on the left into a cone shape, as if making a water cup from a piece of paper. We'll give the region on the left the usual Euclidean metric. But we'll describe this metric with polar coordinates, r and θ.

When we map this region to the cone in R^3, we don't want to use the previous map $(r\cos\theta, r\sin\theta, r)$. Instead we must use a map of the form $(ar\cos b\theta, ar\sin b\theta, ar)$. Indeed, b must be there so that when we only go around the portion of the pie remaining after someone ate a piece, we'll go clear around the cone, and a must be there because distances along the radial r go up the side of the cone, rather than just horizontally out in R^3, and thus are longer. By using some high school geometry, we could figure out a and b and you may have done that in the exercises. I'll work backwards and figure out a and b to make the geometry on the left be Euclidean.

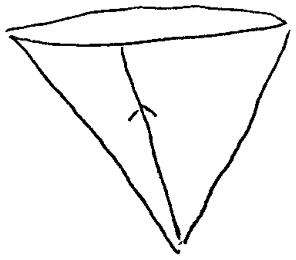




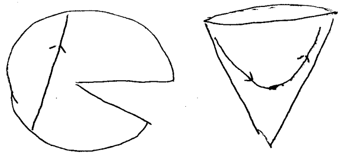
The derivative of our map with respect to r is $(a\cos b\theta, a\sin b\theta, a)$ and the derivative with respect to θ is $(-abr\sin b\theta, abr\cos b\theta, 0)$. Taking dot products, we get $g_{11} = a^2 + a^2, g_{12} = 0, g_{22} = a^2b^2r^2$. If $2a^2 = 1$, i.e., $a = \frac{1}{\sqrt{2}}$, and $a^2b^2 = a$, i.e., $b = \sqrt{2}$, then $g_{11} = 1, g_{12} = 0, g_{22} = r^2$ and so the metric tensor is $ds^2 = dr^2 + r^2 d\theta^2$, which equals the polar form of the Euclidean metric. So the piece on the left will have a Euclidean metric in rectangular coordinates. Therefore geodesics on the left will be ordinary straight lines.

Notice that we want $0 \leq b\theta \leq 2\pi$, and so $0 \leq \theta \leq \frac{2\pi}{b} = \frac{2\pi}{\sqrt{2}} = \sqrt{2}\pi = 1.707\pi$. So the boundary of the figure on the left is $\frac{1.707}{2} = .707 = 71\%$ of the distance around the complete circle.

12: We easily deduce the form of geodesics on the cone. Notice that straight lines through the origin correspond to straight lines which start at the tip of the cone.



In all other cases, lines as on the left correspond to curves which have a point closest to the tip and then curve outward, as shown on the picture below.



These curves do not circle completely around the cone. Indeed, by drawing a line on the left very close to the center, we get a curve which hits the boundary circle of radius one at points about π apart. The distance clear around the cone is $\sqrt{2}\pi$, so the geodesic ends up about $\frac{\pi}{\sqrt{2}\pi} = \frac{1}{\sqrt{2}} = .707 = 71\%$ of the distance around the cone.