Solutions #5, April 29, 2005

1: Let $\gamma(t) = (t, 0)$. Then $\gamma'(t) = (1, 0)$ and $||\gamma'(t)||^2 = g_{11} = \frac{4}{(1-t^2)^2}$. So the length is

$$\int_0^r ||\gamma'(t)||\ dt = \int_0^r \frac{2}{1-t^2}\ dt = \int_0^r \left\{\frac{1}{1-t} + \frac{1}{1+t}\right\}\ dt = [-\ln(1-t) + \ln(1+t)]|_0^r = \ln\frac{1+r}{1-r}$$

2: If $d = \ln\frac{1+r}{1-r}$, then $e^d = \frac{1+r}{1-r}$, so $e^d - re^d = 1 + r$ and $e^d - 1 = r(e^d + 1)$. Therefore

$$r = \frac{e^d - 1}{e^d + 1}.$$

3: Now we must change the meaning of t; ignore the previous parameterization of the curve. At time t we will be at a spot $(r, 0)$. This r depends on t in some way. When we compute the distance traveled, we'll get $d = \ln\frac{1+r}{1-r}$ and we want this to actually be t. So $t = \ln\frac{1+r}{1-r}$ and therefore $r = \frac{e^t-1}{e^t+1}$. So the parameterized curve which travels at unit speed is

$$\gamma(t) = \left(\frac{e^t - 1}{e^t + 1}, 0\right)$$

Notice that $\gamma(0) = (0, 0)$. The first component is always smaller than 1 because $e^t - 1 < e^t + 1$. As t approaches infinity, e^t becomes enormously large, and the numerator and denominator are large numbers within two of each other, so their quotient approaches 1. Hence $\lim_{t\to\infty}\gamma(t) = (1, 0)$.

4: We have $g_{11} = \frac{4}{(1-r^2)^2}, g_{12} = 0, g_{22} = \frac{4r^2}{(1-r^2)^2}$. Then

$$\Gamma^k_{ij} = \frac{1}{2}(g^{-1})_{kk}\left(\frac{\partial g_{ik}}{\partial x_j} + \frac{\partial g_{jk}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_k}\right)$$

Using our standard analysis, the only nonzero terms are $\Gamma^1_{11}, \Gamma^1_{22}, \Gamma^2_{12}$.

Then

$$\Gamma^1_{11} = \frac{1}{2}\frac{(1-r^2)^2}{4}\frac{\partial g_{11}}{\partial r} = \frac{1}{2}\frac{(1-r^2)^2}{4}4(-2)(1-r^2)^{-3}(-2r) = \frac{2r}{1-r^2}$$

$$\Gamma^1_{22} = \frac{1}{2}\frac{(1-r^2)^2}{4}\left(-\frac{\partial g_{22}}{\partial r}\right) = -\frac{1}{2}\frac{(1-r^2)^2}{4}\frac{(1+r^2)8r}{(1-r^2)^3} = -\frac{r(1+r^2)}{1-r^2}$$

$$\Gamma^2_{12} = \frac{1}{2}\frac{(1-r^2)^2}{4r^2}\frac{\partial g_{22}}{\partial r} = \frac{1}{2}\frac{(1-r^2)^2}{4r^2}\frac{(1+r^2)8r}{(1-r^2)^3} = \frac{1+r^2}{r(1-r^2)}$$

5:

$$\frac{d^2r}{dt^2} + \frac{2r}{1-r^2}\left(\frac{dr}{dt}\right)^2 - \frac{r(1+r^2)}{1-r^2}\left(\frac{d\theta}{dt}\right)^2 = 0$$

$$\frac{d^2\theta}{dt^2} + 2\frac{1+r^2}{r(1-r^2)}\frac{dr}{dt}\frac{d\theta}{dt} = 0$$

6: If θ is constant, then $\frac{d\theta}{dt} = \frac{d^2\theta}{dt^2} = 0$ and the second equation is trivially true.

7: The remaining equation now reads

$$\frac{d^2r}{dt^2} + \frac{2r}{1-r^2}\left(\frac{dr}{dt}\right)^2 = 0$$

By symmetry, we can suppose that the radial line is the x-axis. So our curve has the form $\gamma(t) = (r(t), 0)$. Notice that $\gamma'(t) = (r'(t), 0)$ and $||\gamma'||^2 = g_{11}(r')^2 = \frac{4}{(1-r^2)^2}(r')^2$. Therefore

$$||\gamma'(t)|| = \frac{2r'(t)}{1-r^2}$$

This length will be constant just in case its derivative with respect to t is zero. The derivative is

$$\frac{(1-r^2)2r'' - 2r'(-2r)r'}{(1-r^2)^2}$$

and the condition that this is zero is

$$(1-r^2)2r'' + 4r(r')^2 = 0$$

which simplifies to

$$\frac{d^2r}{dt^2} + \frac{2r}{1-r^2}\left(\frac{dr}{dt}\right)^2 = 0$$

This is exactly the remaining equation.