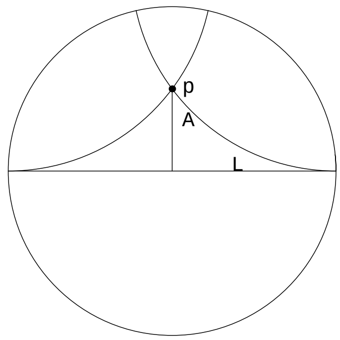
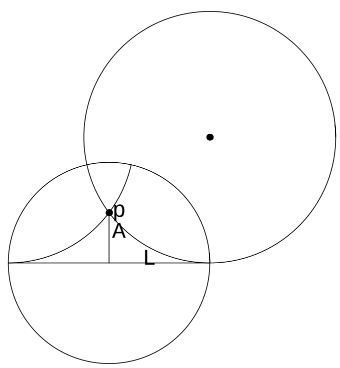
Solutions #6, May 9, 2005

From Previous Exercise Set:

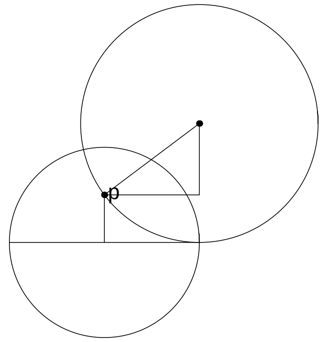
8: Any circle within the angle created by the two "limiting parallels" in the figure below is parallel to the x-axis.



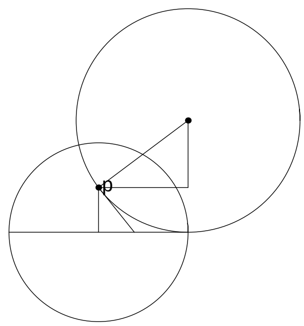
9: Draw the complete right hand limiting parallel as below. This circle meets the boundary of the Poincare disk perpendicularly at $(1, 0)$, so its center must be above this point. Say the center of this circle is $(1, a)$.



Let r be the Euclidean height of the point p above the x-axis. By the Pythagorean theorem applied to the triangle on the next page, $1^2 + (a - r)^2 = a^2$ and so $a = \frac{1+r^2}{2r}$.



The two triangles in the picture below are similar, so $\tan\alpha = \frac{a-r}{1} = \frac{1+r^2}{2r} - r = \frac{1-r^2}{2r}$



By exercise 2, the non Euclidean length d of the perpendicular from p to the x-axis satisfies $r = \frac{e^d-1}{e^d+1}$, so

$$\tan\alpha = \frac{1-\left(\frac{e^d-1}{e^d+1}\right)^2}{2\left(\frac{e^d-1}{e^d+1}\right)} = \frac{(e^d+1)^2-(e^d-1)^2}{2(e^d+1)(e^d-1)} = \frac{4e^d}{2(e^{2d}-1)} = \frac{2}{e^d-e^{-d}}$$

We can get a nicer formula as follows. Multiply top and bottom by e^{-d} to conclude that

$$\tan\alpha = \frac{2e^{-d}}{1-e^{-2d}}$$

On the other hand

$$\tan\alpha = \frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}$$

Consequently,

$$\tan\frac{\alpha}{2} = e^{-d}$$

As $d \to 0$, $\tan\frac{\alpha}{2} \to 1$, so $\frac{\alpha}{2} \to \frac{\pi}{4}$, so $\alpha \to \frac{\pi}{2}$, which is the value of α in Euclidean geometry. As $d \to \infty$, $\tan\frac{\alpha}{2} \to 0$, so $\alpha \to 0$.

From New Exercise Set:

1: We have $g(x,y) = xy^2z = xy^2(x^3y^2) = x^4y^4$. Then

$$X(g) = 2\frac{\partial g}{\partial x} + 3\frac{\partial g}{\partial y} = 2(4x^3y^4) + 3(4x^4y^3) = 8 + 12 = 20.$$

Recall that $X = (a,b)$ corresponds to $\tilde{X} = \left(a, b, a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial x}\right)$. In our case this equals $\tilde{X} = (2, 3, 2(3x^2y^2) + 3(2x^3y)) = (2, 3, 12)$. So

$$\tilde{X}(\tilde{g}) = 2\frac{\partial g}{\partial x} + 3\frac{\partial g}{\partial y} + 12\frac{\partial g}{\partial z} = 2y^2z + 3(2xyz) + 12xy^2 = 20$$

2: We have $[X,Y]f = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) 2\frac{\partial f}{\partial x} - 2\frac{\partial}{\partial x}\left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right) = 2x\frac{\partial^2 f}{\partial x^2} + 2y\frac{\partial^2 f}{\partial x \partial y} - 2\frac{\partial f}{\partial x} - 2x\frac{\partial^2 f}{\partial x^2} - 2y\frac{\partial^2 f}{\partial x \partial y}$ and this equals $-2\frac{\partial f}{\partial x}$. So

$$[X,Y] = -2\frac{\partial}{\partial x} = -Y$$

3: The derivative with respect to X is $(\cos\theta, \sin\theta, 0)$ and the derivative with respect to θ is $(-r\sin\theta, r\cos\theta, 0)$.

The vector field $(x, y, 0)$ points horizontally outward from the cone, and the vectors are longer and longer the higher up we go. Differentiating with respect to r is the same thing as asking how the vectors change as we go radially outward and higher up the cone. These vectors all have the same direction and only increase in length, so the derivative should point in the same direction as the vectors. Moreover, the rate of increase is the same at any height. So the answer is $(\cos\theta, \sin\theta, 0)$, a vector which points outward in the same direction as $(x, y, 0)$ but always has length one.

As we change θ, we move in a circle around the cone. Then the vectors all have the same length, but change in direction. Notice that the *change* points tangent to the circle, as the formula $(-r\sin\theta, r\cos\theta, 0)$ suggests. The higher we go, the faster we circle the cone and so the faster the vectors change, and sure enough the length of this derivative is r and increases with height.

4: The normal points in the direction $\frac{\partial s}{\partial u} \times \frac{\partial s}{\partial v} = (1, 0, v) \times (0, 1, u) = (-v, -u, 1)$ and has length one. So it equals

$$n = \frac{(-v, -u, 1)}{\sqrt{1 + u^2 + v^2}}$$

The derivative of this expression with respect to u is

$$\frac{(0, -1, 0)}{\sqrt{1 + u^2 + v^2}} - \frac{1}{2}\frac{2u}{(1 + u^2 + v^2)^{3/2}}(-v, -u, 1)$$

and at the origin this equals $(0, -1, 0)$. Similarly the derivative with respect to v at the origin is $(-1, 0, 0)$. Both vectors are tangent to the saddle because the saddle is flat at the origin. Notice that the derivative with respect to X points in the $-Y$ direction and the derivative with respect to Y points in the $-X$ direction.

The derivative with respect to $X + Y$ is $-Y - X = -(X + Y)$; this equals -1 times $X + Y$. The derivative with respect to $X - Y$ is $-Y + X = X - Y$; this equals 1 times $X - Y$. Consequently κ_1 and κ_2 are 1 and -1.

5: This calculation is messier. The normal is

$$\frac{\partial s}{\partial r} \times \frac{\partial s}{\partial \theta} = (\cos\theta, \sin\theta, 2r) \times (-r\sin\theta, r\cos\theta, 0) = (-2r^2\cos\theta, -2r^2\sin\theta, r)$$

The length of this vector is $r\sqrt{1 + 4r^2}$ Thus a unit normal is

$$n = \frac{(-2r\cos\theta, -2r\sin\theta, 1)}{\sqrt{1 + 4r^2}}$$

This normal is clearly $\frac{(-2x, -2y, 1)}{\sqrt{1+4(x^2+y^2)}}$.

The derivative of the normal with respect to θ is $\frac{(2r\sin\theta, -2r\cos\theta, 0)}{\sqrt{1+4r^2}}$. Notice that when $Y = \frac{\partial}{\partial\theta}$ we have $\tilde{Y} = \frac{\partial s}{\partial\theta} = (-r\sin\theta, r\cos\theta, 0)$. So

$$Y(n) = \frac{-2}{\sqrt{1 + 4r^2}}Y.$$

It follows that one of the principal curvatures if $\frac{2}{\sqrt{1+r^2}}$ up to sign; the sign depends on a choice of the sign of the normal. This is the curvature when we just change θ.

Similarly the derivative of the normal with respect to r is

$$X(n) = \frac{(-2\cos\theta, -2\sin\theta, 0)}{\sqrt{1 + 4r^2}} - \frac{1}{2}\frac{8r}{(1 + 4r^2)^{3/2}}(-2r\cos\theta, -2r\sin\theta, 1)$$

This can be rewritten

$$\frac{(1 + 4r^2)(-2\cos\theta, -2\sin\theta, 0) - 4r(-2r\cos\theta, -2r\sin\theta, 1)}{(1 + 4r^2)^{3/2}} = \frac{(-2\cos\theta, -2\sin\theta, -4r)}{(1 + 4r^2)^{3/2}}$$

Notice that when $X = \frac{\partial}{\partial r}$ we have $\tilde{X} = \frac{\partial s}{\partial r} = (\cos\theta, \sin\theta, 2r)$. So

$$X(n) = \frac{-2}{(1 + 4r^2)^{3/2}}X$$

It follows that the other principal curvature is $\frac{2}{(1+4r^2)^{3/2}}$. This is the curvature when we just change r.

If we move out along a constant radial line, say the line when $\theta = 0$, our curve is $\gamma(t) = (t, 0, t^2)$. You can easily check that the formula for the curvature of this curve is exactly $\frac{2}{(1+4t^2)^{3/2}}$ obtained above. On the other hand, when we move along a circle by holding r fixed, our curve is $\gamma(t) = (r\cos t, r\sin t, c)$ for some constant c, and the curvature of this circle is not quite $\frac{2}{\sqrt{1+4r^2}}$, although it is close. As we will see later in the course, this difference is related to the fact that the first curve is a geodesic on the surface (although it is not traced with constant speed), while the second is not a geodesic.

Exercises for Graduate Students

6: Let X_i be coordinates with respect to the e_i and $\tilde{X}_i$ be coordinates with respect to the f_i. Then

$$\sum \tilde{X}_i f_i = \sum \tilde{X}_i a_{ji} e_j = \sum \left(\sum a_{ji} \tilde{X}_i \right) e_j = \sum X_j e_j$$

and so

$$X_j = \sum a_{ji} \tilde{X}_i$$

Consequently

$$b(X, Y) = \sum b_{ij} X_i X_j = \sum b_{ij} a_{ik} \tilde{X}_k a_{jl} \tilde{X}_l = \sum (A^T b A)_{kl} \tilde{X}_k \tilde{X}_l$$

So the matrix for b in the new coordinates is $A^T b A$.

Writing $(BX)_i$ for coordinates of $B(X)$ in the old coordinates and $(\tilde{BX})_i$ for coordinates in the new coordinates, we have

$$(BX)_i = \sum B_{ij} X_j = \sum B_{ij} a_{jk} \tilde{X}_k$$

and so

$$(\tilde{BX})_l = \sum (a^{-1})_{li} (BX)_i = \sum (a^{-1})_{li} B_{ij} a_{jk} \tilde{X}_k$$

and thus the matrix for B in the new coordinates is $A^{-1} B A$.

7: Of course $\det(A^{-1}BA) = \det(A^{-1}) \det B \det A = \det(A^{-1}) \det A \det B = \det B$. There are many counterexamples for b. Here's an easy one. Let b be a binlinear form on R^1 given by $b(e_1, e_1) = 1$ The matrix for b is the identity matrix (1). Choose a new basis $f_1 = 2e_1$. Then $b(f_1, f_1) = b(2e_1, 2e_1) = 4b(e_1, e_1) = 4$. So now the matrix is (4) with a different determinant.

8: In the example given in exercise 7, the matrix (1) has eigenvalue 1 and the matrix (4) has eigenvalue 4.

9: There are many proofs. Here is an easy one. Pick a basis. Then the dual vector space consists of linear transformations from V to R and so of $n \times 1$ matrices. Hence it has the same dimension as V. Now it suffices to show that our map is one-to-one because it would then automatically be onto. So suppose $\varphi_v = 0$. Then $\varphi_v(v) = 0$ and so $< v, v >= 0$. But then $v = 0$.

10: Let $v \in V$. Since $\psi^{-1} \circ \varphi(v) = B(v)$, $\varphi(v) = \psi(B(v))$. Both of these are dual vectors; let us evaluate both on $w \in V$. By definition, $\varphi(v)$ evaluated on w gives $b(v, w)$. By definition, $\psi(B(v))$ evaluated on w gives $< B(v), w >$. So $b(v, w) =< B(v), w >$.

If $e_1, \ldots, e_n$ is an orthonormal basis, $b_{ij} = b(e_i, e_j) =< B(e_i), e_j >=< \sum B_{ki} e_k, e_j >= B_{ji}$.

11: If b is symmetric, $< B(X), Y >= b(X, Y) = b(Y, X) =< B(Y), X >=< X, B(Y) >$. This argument is easily reversed. Now assume that $e_1, \ldots, e_n$ is an orthonormal basis. Then $< B(e_i), e_j >=< \sum B_{ki} e_k, e_j >= B_{ji}$ and $< e_i, B(e_j) >=< e_i, \sum B_{kj} e_k >= B_{ij}$.

12: Since the basis vectors are orthonormal, $b_{ij} = B_{ji}$. Since they are eigenvectors, B is a diagonal matrix with entries $\kappa_1, \ldots, \kappa_n$. So b is the same diagonal matrix, and thus $b(X, X) = \sum \kappa_i X_i^2$.