Solutions #7, May 16, 2005

1. Suppose we replace X by λX. Then $b(\lambda X, \lambda X) = \lambda b(X, \lambda X) = \lambda^2 b(X, X)$ and $< \lambda X, \lambda X >= \lambda^2 < X, X >$. So in the quotient λ^2 cancels out and
$$\frac{b(X,X)}{< X, X >} = \frac{b(\lambda X, \lambda X)}{< \lambda X, \lambda X >}$$

2. Let $X = \cos\theta \ e_1 + \sin\theta \ e_2$ in the previous expression. Then
$$b(X,X) = \cos^2\theta \ b(e_1,e_1) = \cos\theta\sin\theta \ b(e_1,e_2) + \sin\theta\cos\theta \ b(e_2,e_1) + \sin^2\theta \ b(e_2,e_2)$$
However $b(e_i, e_j) = - < B(e_i), e_j >= - < -\kappa_i e_i, e_j >= \kappa_i \delta_{ij}$. So
$$b(X,X) = \cos^2\theta \ \kappa_1 + \sin^2\theta \ \kappa_2$$
In exactly the same way,
$$< X, X >= \cos^2\theta \ < e_1, e_1 > + \cos\theta\sin\theta \ < e_1, e_2 > + \sin\theta\cos\theta \ < e_2, e_1 > + \sin^2\theta \ < e_2, e_2 >$$
and this becomes
$$< X, X >= \cos^2\theta + \sin^2\theta = 1$$
Hence $K(X) = \cos^2\theta \ \kappa_1 + \sin^2\theta \ \kappa_2$. We can rewrite this as $\cos^2\theta \ \kappa_1 + (1 - \cos^2\theta) \ \kappa_2 = \cos^2\theta \ (\kappa_1 - \kappa_2) + \kappa_2$.
However, $\cos 2\theta = 2\cos^2\theta - 1$, so $\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$. The formula for $K(X)$ can be rewritten using this formula as
$$K(X) = \frac{1}{2}(\cos 2\theta + 1)(\kappa_1 - \kappa_2) + \kappa_2 = \cos 2\theta \ \frac{\kappa_1 - \kappa_2}{2} + \frac{\kappa_1 + \kappa_2}{2}$$
Using Mathematica, the command

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Plot[Cos[2 t] (3 - 2)/2 + (3 + 2)/2, {t, 0, 2 Pi}, PlotRange -> {0, 3}]
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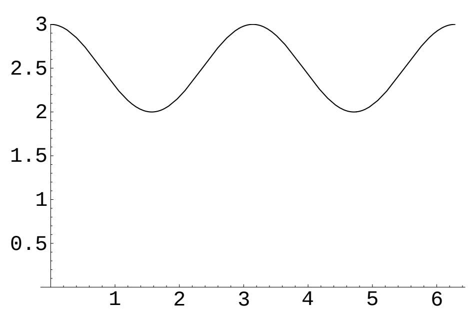
gives



3. Unfortunately, there is a minor misprint in this problem. The eigenvalues of B are actually $\kappa_1 = \frac{2}{(1+4r^2)^{3/2}}$ and $\kappa_2 = \frac{2}{\sqrt{1+4r^2}}$. So there is a missing 2 in the formula for κ_1 in the problem.

   If we turn the paraboloid over so it points along the y-axis, then it becomes a surface of revolution. Specifically it is the surface obtained by revolving $y = f(x) = \sqrt{x}$ about the x-axis. This surface is parametrized by $(x, \theta) \to (x, \sqrt{x}\cos\theta, \sqrt{x}\sin\theta)$ and the curves with constant angle θ are geodesics which travel horizontally along the surface. Tilting the surface back up to its original position, these curves become curves like $t \to (t, 0, t^2)$.

   The curvature of this curve is

$$\kappa = \frac{||(1,0,2t) \times (0,0,2)||}{||(1,0,2t)||^{3/2}} = \frac{||(0,-2,0)||}{(1+4t^2)^{3/2}} = \frac{2}{(1+4t^2)^{3/2}}$$

   Notice that $\frac{2}{\sqrt{1+4r^2}}$ is slightly smaller than $\frac{2}{\sqrt{4r^2}} = \frac{1}{r}$. The geodesic which touches this circle must be in the process is "bouncing back", so it comes from above, touches the circle, and then goes back up. Thus it is approximated by a slightly larger circle.

4. By page 83 of the lecture notes, the κ_i for the surface obtained by revolving $y = f(x)$ about the x-axis are $\kappa_1 = \frac{f''}{(1+(f')^2)^{3/2}}$ and $\kappa_2 = \frac{-1}{f(1+(f')^2)^{1/2}}$. We can get a torus by revolving the function $f(x) = R \pm \sqrt{r^2 - x^2}$ about the x-axis. Here the minus sign gives the inner part of the torus, and the plus sign gives the outer part of the torus.

   Notice that $f' = \pm\frac{-x}{\sqrt{r^2-x^2}}$. Then $f'' = \pm\frac{-r^2}{(r^2-x^2)^{3/2}}$. Then $\sqrt{1+(f')^2} = \frac{r}{\sqrt{r^2-x^2}}$. So

$$\kappa_1 = \pm\frac{-r^2}{(r^2-x^2)^{3/2}} \cdot \frac{(r^2-x^2)^{3/2}}{r^3} = \pm\frac{-1}{r}$$

   and

$$\kappa_2 = \frac{-1}{\left(R \pm \sqrt{r^2-x^2}\right)\left(\frac{r}{\sqrt{r^2-x^2}}\right)} = -\frac{1}{\left(\frac{Rr}{\sqrt{r^2-x^2}} \pm r\right)}$$

   In the lecture notes on page 83, we assumes that the normal pointed outward from the surface of revolution. This choice means that the normal on the outer portion of the torus points out of the torus, but the normal on the inner portion of the torus points into the solid portion of the torus. This yields a discontinuous normal. Let us choose the normal which points out of the solid portion of the torus. Thus the two curvatures above should be taken with the given sign when $\pm$ equals $+$, but with opposite signs when $\pm$ equals $-$. Thus on the outer portion of the torus the κ_i are

$$\kappa_1 = -\frac{1}{r} \qquad \text{and} \qquad \kappa_2 = -\frac{1}{\left(\frac{Rr}{\sqrt{r^2-x^2}} + r\right)}$$

   and on the inner portion of the torus the κ_i are

$$\kappa_1 = -\frac{1}{r} \qquad \text{and} \qquad \kappa_2 = \frac{1}{\left(\frac{Rr}{\sqrt{r^2-x^2}} - r\right)}$$

One of these curvatures is $\frac{1}{r}$ up to sign, as predicted. The second curvature at $x = 0$ where we are on the inside or outside of the large circle is $\frac{-1}{R+r}$ and $\frac{1}{R-r}$, again as predicted. Notice that the κ_i have opposite signs on the inside of the torus, and the same signs on the outside. So the torus is saddle-like on the inside half, and paraboloid-like on the outside half.

Finally, when $x = r$ the curvatures for κ_2 are zero, so the torus looks like a watering trough along the circles which divide saddle-like behavior from paraboloid-like behavior.

5. Using real coordinates, the map to R^4 is

$$(\phi, \theta) \to (\cos\phi, \sin\phi, \cos\theta, \sin\theta)$$

Hence

$$\frac{\partial s}{\partial \phi} = (-\sin\phi, \cos\phi, 0, 0)$$

$$\frac{\partial s}{\partial \theta} = (0, 0, -\sin\theta, \cos\theta)$$

and so $g_{11} = ||(-\sin\phi, \cos\phi, 0, 0)||^2 = 1, g_{12} = 0, g_{22} = 1$.

6. Consider the function f on a torus embedded in R^3 given by

$$f(p) = (\text{distance from } p \text{ to the origin})^2$$

Since the torus is compact, this function has a maximum, M, at some point p_0. The torus is completely inside the sphere of radius M about the origin, and tangent to this sphere at p_0. If we rotate and translate the torus so p_0 becomes the origin and the tangent plane to the sphere at p_0 becomes the xy-plane, then the torus will be above a sphere of radius M through p_0 with center at $(0, 0, M)$. The curvatures of this sphere are all $\frac{1}{M}$; clearly the curvatures of the torus must be larger than this. Hence they are both nonzero, so the Gaussian curvature at p_0 is nonzero. It is actually positive there. Hence the geometry on the torus near p_0 cannot be Euclidean.

7. In this problem I was thinking only of oriented surfaces. The truth is that the torus and the Klein bottle are the only compact surfaces which can be embedded in a high dimensional R^n such that the induced geometry is Euclidean. Consult the web for an appropriate embedding of the Klein bottle into R^5.

   If the geometry were flat, we could cut up the surface into small Euclidean triangles with geodesic sides, i.e., straight sides. The sum of the angles of each such triangle would be π.

   If we sum up the angles of all triangles in this subdivision, we get

$$\pi(\text{number of triangles})$$

   But we can perform this sum a second way, but summing the triangular angles at each vertex, and then sum over all vertices. So this number is also

$$2\pi(\text{number of vertices})$$

Consequently,

$$(\text{number of triangles}) = 2(\text{number of vertices})$$

Each edge is an edge of two triangles, so if we count the number of edges of a triangle and then sum over all triangles, we get

$$3(\text{number of triangles}) = 2(\text{number of edges})$$

Hence the Euler characteristic of the surface is

$$(\text{number of vertices}) - (\text{number of edges}) + (\text{number of triangles})$$

$$= \frac{1}{2}(\text{number of triangles}) - \frac{3}{2}(\text{number of triangles}) + (\text{number of triangles}) = 0$$

But the only connected compact surfaces with Euler characteristic zero are the torus and the Klein bottle.