Solutions #8, May 23, 2005

1. If we pull U back to coordinates, then $U = \Big(U_1(u,v), U_2(u,v), U_3(u,v)\Big)$.

   Write $\gamma(t) = \Big(u(t), v(t)\Big)$. Then $\gamma' = \left(\frac{du}{dt}, \frac{dv}{dt}\right) = \frac{du}{dt}\frac{\partial}{\partial u} + \frac{dv}{dt}\frac{\partial}{\partial v}$. So

$$\gamma'(t)\Big(U\Big) = \left(\frac{du}{dt}\frac{\partial}{\partial u} + \frac{dv}{dt}\frac{\partial}{\partial v}\right)\Big(\Big(U_1(u,v), U_2(u,v), U_3(u,v)\Big)$$

$$= \left(\frac{\partial U_1}{\partial u}\frac{du}{dt} + \frac{\partial U_1}{\partial v}\frac{dv}{dt}, \frac{\partial U_2}{\partial u}\frac{du}{dt} + \frac{\partial U_2}{\partial v}\frac{dv}{dt}, \frac{\partial U_3}{\partial u}\frac{du}{dt} + \frac{\partial U_3}{\partial v}\frac{dv}{dt}\right)$$

$$= \text{ (by the chain rule) } \left(\frac{dU_1}{dt}, \frac{dU_2}{dt}, \frac{dU_3}{dt}\right) = \frac{dU}{dt}$$

2. We have $X(Y) = b(X,Y)n + \nabla_X Y$. As a special case,

$$\frac{dY}{dt} = \gamma'(t)\Big(Y\Big) = b(\gamma', Y)n + \nabla_{\gamma'} Y$$

   It suffices to prove that the last term is given by the formula in the exercise set.

   In general we have

$$\nabla_X Y = \sum_i X_i \nabla_{\frac{\partial}{\partial u_i}} Y = \sum_i X_i \left[\left(\frac{\partial Y_1}{\partial u_i}, \frac{\partial Y_2}{\partial u_i}\right) + \Big(\Gamma^1_{i1}Y_1 + \Gamma^1_{i2}Y_2, \Gamma^2_{i1}Y_1 + \Gamma^2_{i2}Y_2\Big)\right]$$

$$= \sum_i X_i \left[\left(\frac{\partial Y_1}{\partial u_i}, \frac{\partial Y_2}{\partial u_i}\right) + \Big(\sum_j \Gamma^1_{ij}Y_j, \sum_j \Gamma^2_{ij}Y_j\Big)\right]$$

   In our particular case

$$\nabla_{\gamma'} Y = \sum_i \frac{d\gamma_i}{dt}\left[\left(\frac{\partial Y_1}{\partial u_i}, \frac{\partial Y_2}{\partial u_i}\right) + \Big(\sum_j \Gamma^1_{ij}Y_j, \sum_j \Gamma^2_{ij}Y_j\Big)\right]$$

$$= \left(\sum_i \frac{\partial Y_1}{\partial u_i}\frac{du_i}{dt}, \sum_i \frac{\partial Y_2}{\partial u_i}\frac{du_i}{dt}\right) + \Big(\sum_{ij} \Gamma^1_{ij}\frac{d\gamma_i}{dt}Y_j, \sum_{ij} \Gamma^2_{ij}\frac{d\gamma_i}{dt}Y_j\Big)$$

   In this last expression, $\frac{du_i}{dt}$ and $\frac{d\gamma_i}{dt}$ are the same; I wrote $\frac{du_i}{dt}$ to suggest use of the chain rule. So

$$\nabla_{\gamma'} Y = \left(\frac{dY_1}{dt}, \frac{dY_2}{dt}\right) + \Big(\sum_{ij} \Gamma^1_{ij}\frac{d\gamma_i}{dt}Y_j, \sum_{ij} \Gamma^2_{ij}\frac{d\gamma_i}{dt}Y_j\Big)$$

   and this is the formula in the exercise.

3. The kth component of $\frac{D\gamma'}{dt}$ can be obtained by replacing Y_k by $\frac{d\gamma_k}{dt}$ in the previous equation. Thus it is

$$\frac{d^2\gamma_k}{dt^2} + \sum_{ij} \Gamma^k_{ij}\frac{d\gamma_i}{dt}\frac{d\gamma_j}{dt}$$

   and this is exactly the expression which equals zero by the geodesic equation.

4. Since $\gamma(t)$ has constant speed one, it is parameterized by arc length. So $T = \gamma'$ and

$$\kappa N = \frac{dT}{dt} = \frac{d\gamma'}{dt} = \text{ (by exercise two) } b(\gamma', \gamma')n + \nabla_{\gamma'}\gamma' = b(\gamma', \gamma')n + \frac{D\gamma'}{dt}$$

Since γ is a geodesic, $\frac{D\gamma'}{dt} = 0$ by exercise three. So

$$\kappa N = b(\gamma', \gamma')n$$

Since N and n are both unit vectors, $\kappa = b(\gamma', \gamma')$ and $N = n$. Incidentally, $b(\gamma', \gamma')$ can be negative, as we saw in the previous exercise set, so to make this formula hold we must allow κ to be negative.

5. Suppose $\gamma(t)$ is a geodesic, not necessarily with speed one. Certainly γ has constant speed; call this speed λ. Then $\tau(t) = \gamma\left(\frac{t}{\lambda}\right)$ is a parametrization by arc length because $||\tau'|| = ||\gamma' \frac{1}{\lambda}|| = 1$. By the previous exercise, the curvature of τ is $b(\tau', \tau') = b\left(\frac{\gamma'}{\lambda}, \frac{\gamma'}{\lambda}\right) = \frac{1}{\lambda^2} b(\gamma', \gamma')$. Curvature is a geometric notion independent of parametrization, so this expression is also the curvature of γ. To repeat for emphasis, the curvature of γ is

$$\kappa = \frac{b(\gamma', \gamma')}{\lambda^2} = \frac{b(\gamma', \gamma')}{||\gamma'||^2} = \frac{b(\gamma', \gamma')}{< \gamma', \gamma' >}$$

The exercise immediately follows from this formula.