Solutions #9, June 1, 2005

1. Hello world. This does seem to be working just fine. Wow. Now we are going to see if the system continues working as we continue to type. Yes, it does. Gosh.

2. Notice that $g_{ij} = \frac{\partial s}{\partial u_i} \cdot \frac{\partial s}{\partial u_j}$. If we multiply each s by M, this expression is multiplied by M^2. Recall that

$$\Gamma_{ij}^k = \frac{1}{2} g_{kl}^{-1} \left\{ \frac{\partial g_{il}}{\partial u_j} + \frac{\partial g_{jl}}{\partial u_i} - \frac{\partial g_{ij}}{\partial u_l} \right\}$$

The new g_{ij} are the old ones multiplied by M^2, so the expression inside the curly brackets increases by M^2. On the other hand, the entries of the new g_{ij}^{-1} equal the entries of the old expression divided by M^2, since

$$\left(M^2 A \right)^{-1} = \frac{1}{M^2} A^{-1}$$

So the $\frac{1}{M^2}$ from the inverse matrix cancels the M^2 from the expression inside the curly brackets.

It follows that $\nabla_X Y$ does not change, since the formula for this expression involves only the X_i, the Y_j and their derivatives, and the Γ_{ij}^k. Now

$$R(X, Y, Z, W) = < \nabla_X \nabla_Y Z - \nabla_y \nabla_X Z - \nabla_{[X,Y]} Z, W >$$

and nothing in this formula changes except the outer inner product, which increases by M^2.

On the original surface we choose an orthonormal basis f_1 and f_2. Then

$$R(f_1, f_2, f_1, f_2) = -\kappa_1 \kappa_2.$$

On the new surface, we must replace each f_i by $\frac{f_i}{M}$ and then apply the new R which equals M^2 times the old R. The result will be the old value multiplies by $\frac{M^2}{M^4} = \frac{1}{M^2}$. So the new $-\kappa_1 \kappa_2$ is the old one divided by M^2.

3. In the above formula for Γ_{ij}^k, all partial derivatives of g_{ij} are zero, so $\Gamma_{ij}^k = 0$. Hence the "fudge factor" terms in the formula for $\nabla_{\frac{\partial}{\partial u_i}} Y$ vanish and

$$\nabla_{\frac{\partial}{\partial u_i}} Y = \left(\frac{\partial Y_1}{\partial u_i}, \frac{\partial Y_2}{\partial u_i} \right)$$

Consequently

$$\nabla_{\frac{\partial}{\partial u_i}} \nabla_{\frac{\partial}{\partial u_j}} Y = \left(\frac{\partial^2 Y_1}{\partial u_i \partial u_j}, \frac{\partial^2 Y_2}{\partial u_i \partial u_j} \right)$$

Let $e_i = \frac{\partial}{\partial u_i}$. Clearly $[e_i, e_j] = 0$ by the equality of mixed partial derivatives. The same mixed partial equality shows that $\nabla_{e_1} \nabla_{e_2} Y - \nabla_{e_2} \nabla_{e_1} Y - \nabla_{[e_1, e_2]} Y = 0$ and so $R(e_1, e_2, Y, Z) = 0$ for any Y and Z. In particular $R_{1212} = R(e_1, e_2, e_1, e_2) = 0$. So all $R_{ijkl} = 0$ and thus $\kappa = 0$.

4. According to the cheat sheet, $g_{11} = \sin^2\phi, g_{12} = 0, g_{22} = 1$. So $\det(g_{ij}) = \sin^2\phi$. By theorems in class, $R(f_1, f_2, f_1, f_2) = -\kappa_1\kappa_2$ and $R(e_1, e_2, e_1, e_2) = \det(g_{ij})R(f_1, f_2, f_1, f_2)$, so

$$R(e_1, e_2, e_1, e_2) = -(\sin^2\phi)\ \kappa_1\kappa_2$$

Notice that $[e_1, e_2] = \left[\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi}\right] = 0$ by the equality of mixed partial derivatives, so

$$R(e_1, e_2, e_1, e_2) =< \nabla_{e_1}\nabla_{e_2}e_1 - \nabla_{e_2}\nabla_{e_1}e_1, e_2 >$$

By the last page of the cheat sheet, $\nabla_{e_1}e_1 = -\sin\phi\cos\phi\ e_2$ and $\nabla_{e_2}e_1 = \frac{\cos\phi}{\sin\phi}\ e_1$ and so

$$R(e_1, e_2, e_1, e_2) =< \nabla_{e_1}\left(\frac{\cos\phi}{\sin\phi}\ e_1\right) - \nabla_{e_2}\left(-\sin\phi\cos\phi\ e_2\right), e_2 >= -(\sin^2\phi)\ \kappa_1\kappa_2$$

and therefore

$$< \nabla_{e_1}\left(\frac{\cos\phi}{\sin\phi}\ e_1\right), e_2 > + < -\nabla_{e_2}\left(-\sin\phi\cos\phi\ e_2\right), e_2 >= -(\sin^2\phi)\ \kappa_1\kappa_2$$

Consider the first term $\nabla_{e_1}\left(\frac{\cos\phi}{\sin\phi}\ e_1\right)$. Since $e_1 = \frac{\partial}{\partial\theta}$, the derivative of $\frac{\cos\phi}{\sin\phi}$ by e_1 is zero, so $\nabla_{e_1}\left(\frac{\cos\phi}{\sin\phi}\ e_1\right) = \frac{\cos\phi}{\sin\phi}\ \nabla_{e_1}e_1 = \frac{\cos\phi}{\sin\phi}\ \left(-\sin\phi\cos\phi\ e_2\right) = -\cos^2\phi\ e_2$. When we dot this term with e_2, we get $-\cos^2\phi$.

Consider the second term $-\nabla_{e_2}\left(-\sin\phi\cos\phi\ e_2\right)$. Since $e_2 = \frac{\partial}{\partial\phi}$, this expression equals $\frac{\partial}{\partial\phi}(\sin\phi\cos\phi)\ e_2 + \sin\phi\cos\phi\ \nabla_{e_2}e_2$. By the cheat sheet, $\nabla_{e_2}e_2 = 0$, so this term equals $\left(\cos^2\phi - \sin^2\phi\right)\ e_2$ and its inner product with e_2 is $\left(\cos^2\phi - \sin^2\phi\right)$.

Therefore the last displayed formula gives

$$(-\cos^2\phi) + (\cos^2\phi - \sin^2\phi) = -\sin^2\phi\ \kappa_1\kappa 2$$

and therefore $\kappa_1\kappa_2 = 1$.

If we magnify the unit sphere by R, we get a sphere with radius R. By exercise one, the Gaussian curvature is then divided by R^2, so the Gaussian curvature is $\frac{1}{R^2}$.

5. Notice that $\det(g_{ij}) = \frac{16r^2}{(1-r^2)^4}$, so the argument at the start of the previous exercise now gives

$$R(e_1, e_2, e_1, e_2) = \frac{16r^2}{(1-r^2)^4}R(f_1, f_2, f_1, f_2) = \frac{16r^2}{(1-r^2)^4}(-\kappa)$$

By equality of mixed partials, $[e_1, e_2] = 0$, so

$$< \nabla_{e_1}\nabla)e_2e_1 - \nabla_{e_2}\nabla_{e_1}e_1, e_2 >= -\frac{16r^2}{(1-r^2)^4}\kappa$$

But $\nabla_{e_1} e_1 = \Gamma^1_{11} e_1 + \Gamma^2_{11} e_2$ and $\nabla_{e_2} e_1 = \Gamma^1_{12} e_1 + \Gamma^2_{12} e_2$. So

$$< \nabla_{e_1} \left(\Gamma^1_{12} e_1 + \Gamma^2_{12} e_2 \right), e_2 > - < \nabla_{e_2} \left(\Gamma^1_{11} e_1 + \Gamma^2_{11} e_2 \right), e_2 >= -\frac{16r^2}{(1-r^2)^4} \kappa$$

Notice that the g_{ij} depend only on the first variable r and $g_{12} = 0$. So if $\Gamma^k_{ij} \neq 0$, then one of i, j, k must be one and the other two must be equal. So the only nonzero Γ^k_{ij} are $\Gamma^1_{11}, \Gamma^1_{22}$, and Γ^2_{12}. It follows that the displayed formula simplifies to

$$< \nabla_{e_1} \left(\Gamma^2_{12} e_2 \right), e_2 > - < \nabla_{e_2} \left(\Gamma^1_{11} e_1 \right), e_2 >= -\frac{16r^2}{(1-r^2)^4} \kappa$$

In this expression, the derivative of Γ^1_{11} with respect to $e_2 = \frac{\partial}{\partial \theta}$ is zero because Γ^1_{11} depends only on r. So the above formula simplifies to

$$\frac{\partial \Gamma^2_{12}}{\partial r} < e_2, e_2 > + \Gamma^2_{12} < \nabla_{e_1} e_2, e_2 > - \Gamma^1_{11} < \nabla_{e_2} e_1, e_2 >= -\frac{16r^2}{(1-r^2)^4} \kappa$$

In the first term, $< e_2, e_2 >= \frac{4r^2}{(1-r^2)^2}$. Since $[e_1, e_2] = 0$ because mixed partial derivatives are equal, $\nabla_{e_1} e_2 = \nabla_{e_2} e_1$. Consequently the key portion of the second and third terms are equal. Moreover, $< e_1, e_2 >= g_{12} = 0$, so these terms are

$$< \nabla_{e_2} e_1, e_2 >=< \nabla_{e_1} e_2, e_2 >=< \Gamma^1_{12} e_1 + \Gamma^2_{12} e_2, e_2 >= \Gamma^2_{12} \frac{4r^2}{(1-r^2)^2}$$

Putting all three terms together, we have

$$\frac{\partial \Gamma^2_{12}}{\partial r} \frac{4r^2}{(1-r^2)^2} + \left(\Gamma^2_{12} \right)^2 \frac{4r^2}{(1-r^2)^2} - \Gamma^1_{11} \Gamma^2_{12} \frac{4r^2}{(1-r^2)^2} = -\frac{16r^2}{(1-r^2)^4} \kappa$$

Cancelling $\frac{4r^2}{(1-r^2)^2}$, we obtain

$$\frac{\partial \Gamma^2_{12}}{\partial r} + \left(\Gamma^2_{12} \right)^2 - \Gamma^1_{11} \Gamma^2_{12} = -\frac{4}{(1-r^2)^2} \kappa$$

Now

$$\Gamma^2_{12} = \frac{1}{2} \frac{(1-r^2)^2}{4r^2} \left\{ \frac{\partial}{\partial r} \frac{4r^2}{(1-r^2)^2} \right\} = \frac{1+r^2}{r(1-r^2)}$$

and

$$\Gamma^1_{11} = \frac{1}{2} \frac{(1-r^2)^2}{4} \left\{ \frac{\partial}{\partial r} \frac{4}{(1-r^2)^2} \right\} = \frac{2r}{(1-r^2)}$$

Consequently

$$\frac{\partial}{\partial r} \left(\frac{1+r^2}{r(1-r^2)} \right) + \frac{(1+r^2)^2}{r^2(1-r^2)^2} - \frac{2(1+r^2)}{(1-r^2)^2} = -\frac{4}{(1-r^2)^2} \kappa$$

A brief calculation shows that this expression is the same as

$$\frac{4}{(1-r^2)^2} = -\frac{4}{(1-r^2)^2}\kappa$$

which implies that $\kappa = -1$.

6. Notice that $e_2 = \frac{\partial}{\partial v}$ is the tangent vector of the geodesic in the v direction. We proved in a previous exercise that the geodesic equation can be written $\nabla_{\gamma'}\gamma' = 0$, which we also wrote as $\frac{D}{dt}\gamma' = 0$. Consequently $\nabla_{e_2} e_2 = 0$. Our geodesic has constant speed and thus its derivative has constant length. Since we defined the geodesic in the v direction to have speed 1, we have $< e_2, e_2 >= g_{22} = 1$.

   The u-geodesic only exists when $v = 0$. On this u-geodesic we drew v-geodesics in a perpendicular direction. The tangent vector to the u-geodesic is e_1 and the tangent vectors to the v-geodesics are e_2. It follows that $< e_1, e_2 >= 0$ on points of the form $(u, 0)$.

   On the other hand, the derivative of $< e_1, e_2 >$ in the v direction is

$$e_2 < e_1, e_2 >=< \nabla_{e_2} e_1, e_2 > + < e_1, \nabla_{e_2} e_2 > .$$

   Since $\nabla_{e_2} e_2 = 0$ everywhere, this gives

$$e_2 < e_1, e_2 >=< \nabla_{e_2} e_1, e_2 > .$$

   Now e_1 is partial differentiation in the u direction and e_2 is partial differential in the v direction; these commute by equality of mixed partial derivatives. So $[e_1, e_2] = 0$ and $\nabla_{e_2} e_1 = \nabla_{e_1} e_2$. So

$$e_2 < e_1, e_2 >=< \nabla_{e_2} e_1, e_2 > . =< \nabla_{e_1} e_2, e_2 > . = \frac{1}{2} e_1 < e_2, e_2 >= \frac{1}{2} e_1(1) = 0$$

   Said another way, the derivative of $< e_1, e_2 >$ in the v direction is zero, so $< e_1, e_2 >$ is constant in the v direction. Since this expression is zero at points of the form $(u, 0)$, it is always zero.

   We conclude that $g_{12} = 0$ and $g_{22} = 1$. It follows that g_{11} is some positive number which depends on u and v. Write $g_{11} = h^2(u, v)$.

   We are now going to compute κ as a function of h. Notice that

$$R(e_1, e_2, e_1, e_2) = \det(g_{ij}) R(f_1, f_2, f_1, f_2) = -\det(g_{ij}\kappa = -h^2\kappa$$

   Since e_1 and e_2 are commuting partial derivatives, $[e_1, e_2] = 0$ and thus

$$R(e_1, e_2, e_1, e_2) =< \nabla_{e_1}\nabla_{e_2} e_1 - \nabla_{e_2}\nabla_{e_1} e_1, e_2 >= -h^2\kappa$$

   Thus

$$< \nabla_{e_1}\left(\Gamma_{12}^1 e_1 + \Gamma_{12}^2 e_2\right), e_2 > - < \nabla_{e_2}\left(\Gamma_{11}^1 e_1 + \Gamma_{11}^2 e_2\right), e_2 >= -h^2\kappa$$

   The only nonconstant g_{ij} is g_{11} so Γ_{ij}^k can only be nonzero if two of the terms equal one. Thus the only nonzero terms are $\Gamma_{11}^1, \Gamma_{12}^1, \Gamma_{11}^2$. So the previous formula simplifies to

$$< \nabla_{e_1}\left(\Gamma_{12}^1 e_1\right), e_2 > - < \nabla_{e_2}\left(\Gamma_{11}^1 e_1 + \Gamma_{11}^2 e_2\right), e_2 >= -h^2\kappa$$

In the first term, if we differentiate Γ_{12}^1 and leave e_1 alone, then we'll have a multiple of $< e_1, e_2 >= 0$. So we must leave Γ_{12}^1 alone and differentiate e_1. Similar arguments show that the previous formula reduces to

$$\Gamma_{12}^1 < \nabla_{e_1} e_1, e_2 > -\Gamma_{11}^1 < \nabla_{e_2} e_1, e_2 > -e_2(\Gamma_{11}^2) < e_2, e_2 > -\Gamma_{11}^2 < \nabla_{e_2} e_2, e_2 >= -h^2\kappa$$

or

$$\Gamma_{12}^1 < \Gamma_{11}^1 e_1 + \Gamma_{11}^2 e_2, e_2 > -\Gamma_{11}^1 < \Gamma_{12}^1 e_1 + \Gamma_{12}^2 e_2, e_2 > -\frac{\partial}{\partial v}(\Gamma_{11}^2) - \Gamma_{11}^2 < \Gamma_{22}^1 e_1 + \Gamma_{22}^2 e_2, e_2 >= -h^2\kappa$$

and so

$$\Gamma_{12}^1 \Gamma_{11}^2 - \Gamma_{11}^1 \Gamma_{12}^2 - \frac{\partial \Gamma_{11}^2}{\partial v} - \Gamma_{11}^2 \Gamma_{22}^2 = -h^2\kappa$$

The only term Γ_{ij}^2 which is nonzero is Γ_{11}^2, so this reduces to

$$\Gamma_{12}^1 \Gamma_{11}^2 - \frac{\partial \Gamma_{11}^2}{\partial v} = -h^2\kappa$$

We must now compute two Christoffel symbols. We have

$$\Gamma_{12}^1 = \frac{1}{2} \frac{1}{h^2} \frac{\partial h^2}{\partial v} = \frac{1}{h} \frac{\partial h}{\partial v}$$

and

$$\Gamma_{11}^2 = \frac{1}{2} \left\{ -\frac{\partial h^2}{\partial v} \right\} = -h \frac{\partial h}{\partial v}$$

Thus

$$\Gamma_{12}^1 \Gamma_{11}^2 - \frac{\partial \Gamma_{11}^2}{\partial v} = -\left(\frac{\partial h}{\partial v} \right)^2 - \frac{\partial}{\partial v} \left(-h \frac{\partial h}{\partial v} \right) = -h^2\kappa$$

or

$$-\left(\frac{\partial h}{\partial v} \right)^2 + \left(\frac{\partial h}{\partial v} \right)^2 + h \frac{\partial^2 h}{\partial v^2} = -h^2\kappa$$

and so

$$\frac{\partial^2 h}{\partial v^2} = -h\kappa$$

Notice that this formula still does not yet the hypothesis that κ is constant.

But now let us assume that κ is constant and indeed that it is equal to $1, 0$, or -1. Thus $\frac{\partial^2 h}{\partial v^2} = -h$ or $\frac{\partial^2 h}{\partial v^2} = 0$ or $\frac{\partial^2 h}{\partial v^2} = h$. These differential equations are easily solved: $h(u,v) = A(u)\cos v + B(u)\sin v$ or $h(u,v) = B(u)v + A(u)$ or $h(u,v) = A(u)\cosh v + B(v)\sinh v$.

But notice that at points of the form $(u, 0)$ we are on the original geodesic where e_1 is the tangent line and has speed one. So $h(u,0)^2 = 1$. It follows that $h(u,0) = 1$ and thus that

$$h(u,v) = \cos v + B(u)\sin v \text{ if } \kappa = 1$$

$$h(u,v) = B(u) + 1 \text{ if } \kappa = 0$$

$$h(u,v) = \cosh v + B(u)\sinh v \text{ if } \kappa = -1$$

To finish the argument, we will prove that $\frac{\partial h}{\partial v}(u, 0) = 0$. It follows that $B(u) = 0$ and thus that

$$h(u, v) = \cos v \text{ if } \kappa = 1$$

$$h(u, v) = 1 \text{ if } \kappa = 0$$

$$h(u, v) = \cosh v \text{ if } \kappa = -1$$

Notice that $e_2 < e_1, e_1 >= \frac{\partial}{\partial v} h^2 = 2h \frac{\partial h}{\partial v}$. This expression is also $2 < \nabla_{e_1} e_2, e_2 >$. Since mixed partial derivatives are equal, $[e_1, e_2] = 0$ and therefore $\nabla_{e_1} e_2 = \nabla_{e_2} e_1$. So

$$2 < \nabla_{e_2} e_1, e_2 >= 2h \frac{\partial h}{\partial v}$$

But $e_1 < e_1, e_2 >= \frac{\partial}{\partial u} 0 = 0$ is also $e_1 < e_1, e_2 >=< \nabla_{e_1} e_1, e_x > + < e_1, \nabla_{e_1} e_2 >$. The initial coordinate curve along $(u, 0)$ is a geodesic, so $\nabla_{e_1} e_1 = 0$ on all points $(u, 0)$. Consequently the first equation in this paragraph implies that $< e_1, \nabla_{e_1} e_2 >= 0$ on points of the form $(u, 0)$. The displayed equation just before the paragraph then implies that $2h \frac{\partial h}{\partial v} = 0$ at points of the form $(u, 0)$. Since $h \neq 0$, it follows that $\frac{\partial h}{\partial v}(u, 0) = 0$.

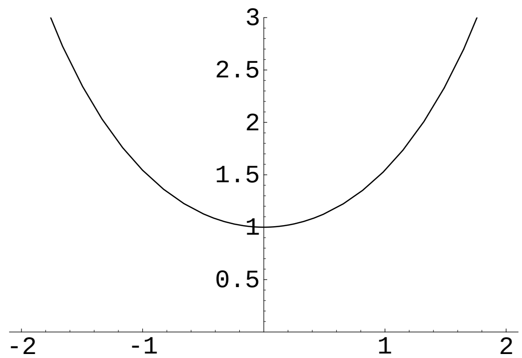
Remark: Finally, let me explain the significance of this last exercise. Suppose two surfaces have the same constant curvature κ. Let p be a point in the first surface and q be a point in the second surface. Then near p on the first surface, we can find coordinates (u, v) as described above; in particular, the g_{ij} are completely determined in this coordinate system. We can find similar coordinates $(\tilde{u}, \tilde{v})$ near q. But then we can map a neighborhood of p to a neighborhood of q by sending (u, v) to $(\tilde{u}, \tilde{v})$. This map preserves distances and angles because both neighborhoods have the same g_{ij}. So the first and second surfaces are locally isometric near p and q.

As a special case, we have proved that when $\kappa = 0$, we can find local coordinates so $g_{ij} = \delta_{ij}$. The converse was proved in exercise one. So a surface is locally Euclidean if and only if its Riemannian curvature tensor is identically zero.

Next consider the case of the sphere. In the last exercise we proved that we can find coordinates (u, v) so $g_{11} = \cos^2(v), g_{12} = 0, g_{22} = 1$. On the spherical cheat sheet we found coordinates (θ, ϕ) such that $g_{11} = \sin^2 \phi, g_{12} = 0, g_{22} = 1$. Why is there a $\cos^2$ in one case and a $\sin^2$ in the other?

Consider again our construction of u and v. Our u geodesic could be taken to be the equator. We can write $u = \theta$ because $(\cos\theta, \sin\theta, 0)$ is this geodesic traveled with speed one. Our v geodesics are lines of constant longitude, so v is essentially ϕ. But this isn't quite right because in spherical coordinates we measure angles down from the north pole rather than up from the equator. So actually $v = \frac{\pi}{2} - \phi$. Therefore $\cos^2 v = \left(\cos\left(\frac{\pi}{2} - \phi\right)\right)^2 = (\sin\phi)^2$. Thus the last exercise actually proves that on any surface with $\kappa = 1$ we can find local coordinates which behave exactly like spherical coordinates.

Finally, when $\kappa = -1$ we obtained $g_{11} = \cosh^2 u$. Here is a graph of the hyperbolic cosine:



Therefore, in non-Euclidean geometry the longitude lines spread apart as we go upward, rather than converging as they do on the sphere. It is easy to see this behavior in the Poincare model: