“Quantization via Mirror Symmetry”

Sergei Gukov
Notes taken by Laura Fredrickson

May 28, 2014
GEAR Junior Retreat (Ann Arbor, MI)

Contents

Abstract

When combined with mirror symmetry, the A-model approach to quantization leads to a fairly simple and tractable problem. The most interesting part of the problem then becomes finding the mirror of the coisotropic brane. We illustrate how it can be addressed in a number of interesting examples related to representation theory and gauge theory, in which mirror geometry is naturally associated with the Langlands dual group. Hyper-holomorphic sheaves and (B, B, B) branes play an important role in the B-model approach to quantization.

The reference for this talk is arXiv:1011.2218

The main goal of these lectures is to explain the word “branes.” Particularly, we’ll focus on branes in the moduli spaces of Higgs bundles, since this particular example is very rich and is useful for illustrating many properties of branes.

The notion of “branes” comes from physics, so the first part will be a gentle colloquium-style introduction to what a brane is to a physicist. After this, we’ll come back and give mathematical definitions.

1 A gentle introduction to branes from the physics perspective

What is a "brane" to a physicist? The beginning of the story is in what physicists call "σ-models". The physics and math communities use different language, but everything is easily translatable. One of the goals of this section is to introduce the physics jargon, so you can fluently speak another language.

1.1 σ-models

A physicist's "**σ-model**" is completely synonymous with the mathematician's problem of studying maps, ϕ, from a Riemann surface Σ into a target manifold X

$$\phi : \Sigma \to X. \tag{1}$$

The surface Σ is called the **source** or **domain** of the σ-model. The space X is called the **target** of the σ-model, and is usually a high-dimensional manifold. Because the domain Σ is two-dimensional, physicists say σ-models are **2d-theories**.

The moduli space $\mathcal{M}$ of all such maps $\phi : \Sigma \to X$ is an interesting geometric object associated to the σ-model. Additional conditions on ϕ in the σ-model are reflected in the moduli space

$$\mathcal{M} = \{\ \phi : \Sigma \to X \mid \text{conditions}\ \}/\sim \tag{2}$$

where $\sim$ is some equivalence relation. Though, ultimately, one may interested in studying either Σ or X, it may be useful to instead study this associated geometric object $\mathcal{M}$. Some of the geometry of the original space can be captured by the geometry of the auxiliary intermediate space $\mathcal{M}$.

▷ **Gromov-Witten Theory:** One interesting mathematical example of a σ-model is Gromov-Witten theory. In Gromov-Witten theory, one is interested in studying a manifold X. To do so, one considers the moduli space of holomorphic maps $\phi : \Sigma \to X$, if necessary modulo certain reparameterizations. One gets a moduli space $\mathcal{M}_\Sigma$

$$\mathcal{M}_\Sigma = \{\phi : \Sigma \to X | \bar{\partial}\phi = 0\}/\sim . \tag{3}$$

associated to each Σ. The geometry of $\mathcal{M}_\Sigma$ captures some of the geometry of X. In particular, in Gromov-Witten theory, the Euler characteristic of the moduli space, $\chi(\mathcal{M}_\Sigma)$ gives an invariant of the space X, known as a Gromov-Witten invariant. Gromov-Witten invariants will continue to appear in the story of the origins of mirror symmetry .◁

Originally, physicists used σ-models to understand Σ. ("What does an observer in Σ see?") However, there's a way to reformulate the story in the perspective of an observer who lives in the high-dimensional target space X. In the early 90s, physics realized this "target space perspective" was helpful. *Why?* People discovered mirror symmetry.

1.2 Mirror Symmetry

Historically, mirror symmetry appeared as a correspondence between two different target manifolds in Gromov-Witten theory.

▷ **Mirror Symmetry and Gromov-Witten Invariants:**
The Gromov-Witten invariants of X, as written, are hard to compute. They involve counting certain stable holomorphic maps. However, people discovered that if the Gromov-Witten invariants were packaged together appropriately, they turn out to be simple invariants associated with the variation of Hodge and complex structures on completely different space $\tilde{X}$.

In Gromov-Witten theory, mirror symmetry first appeared as a relation between two different problems:

1. counting holomorphic maps with symplectic target space (X, ω)

   and

2. computing periodic integrals with complex target space $(\tilde{X}, I)$.

On the counting side, given a symplectic manifold (X, ω), the Gromov-Witten invariants are the Euler characteristics of the moduli spaces $\mathcal{M}_\Sigma$ of holomorphic maps $\phi : \Sigma \to X$, i.e. $\mathrm{GW}(X) = \chi(\mathcal{M}_\Sigma)$. These invariants can be packaged together into various sums. For example, one such sum is

$$\sum_{\substack{g=\text{genus}(\Sigma)\in\mathbb{Z}^{\geq 0}\\ \beta=[\phi(\Sigma)]\in H_2(X)}} \chi(\mathcal{M}_\Sigma) e^{-\int_{\phi(\Sigma)}\omega}. \tag{4}$$

The sum involves a weighted contribution from the invariants coming all the different moduli spaces $\mathcal{M}_\Sigma$. Note that the image $\phi(\Sigma)$ appears in the sum. Since the dimension of Σ is 2, generally the image of Σ is also 2, and it is a meaningful question to ask the homology class $[\phi(\Sigma)] \in H_2(X)$. In the above sum, the symplectic form ω on X appears in the pairing of the 2-form ω and the cohomology class $[\phi(\Sigma)] \in H_2(X)$. This sum is only one of many interesting sums in the story, and the symplectic form appears in different ways in the other sums.

These particular sums of Gromov-Witten invariants on the symplectic side can be computed from simple invariants associated with the variation of Hodge and complex structures on the complex side. These invariants are computed by periodic integrals. Residue techniques make these invariants much simpler to compute! So mirror symmetry appeared first at the enumerative level, giving powerful techniques for computing numerical invariants. This pairing between certain invariants of a symplectic manifold (X, ω) and a certain complex manifold $(\tilde{X}, I)$ that appeared in Gromov-Witten Theory is part of a larger story. $\triangleleft$

More broadly, in the early 90s, people realized that mirror symmetry gives a correspondence between the target spaces of two different σ-models: one symplectic and one complex.

$$\begin{array}{cc} \text{target} & \text{target} \\ (X, \omega) & (\tilde{X}, I) \\ \text{symplectic} & \text{complex.} \end{array}$$

Consequently, mirror symmetry pairs a symplectic manifold X and a complex manifold $\tilde{X}$. We don't have a full dictionary describing what objects have pairs, let alone what those pairs are. But we do know that if X is a Calabi-Yau manifold of dimension $\dim_{\mathbb{C}} X = n$, then $\tilde{X}$ is also a Calabi-Yau manifold of the same dimension $\dim_{\mathbb{C}} \tilde{X} = n$ and roughly of the "same type," though topologically $\tilde{X}$ may be very different! Consequently, we will focus on mirror symmetry for Calabi-Yau manifolds.

Remark If X is not Calabi-Yau, the mirror manifold may be of completely different dimension. For example, the mirror manifold of $\mathbb{CP}^n$ is not an n-manifold

1.3 Mirrors of compact Calabi-Yau manifolds

Here, we give some examples of mirror symmetry in Calabi-Yau manifolds of $\dim_{\mathbb{C}} X = 1, 2, 3$. (Actually, the case of $\dim_{\mathbb{C}} X = 0$–far from trivial– is quite interesting and leads to a very beautiful hierarchy of integrable systems, though to discuss this would take us too far afield.) One of the main messages of the entire course is:

> *Any extra structure on the original target space, X, corresponds to some extra structure on the mirror target space, $\tilde{X}$.*

The low-dimensional Calabi-Yau examples that follow illustrate this principle, and the story when the target space is the moduli space of Higgs bundles is even richer!

$\mathbf{n = 1}$: X is elliptic curve

To be a compact Calabi-Yau manifold, X must be Kähler and have trivial canonical class, $\Omega^n = \Lambda^n \mathcal{T}^*(X)$, the n^{th}-exterior power of the holomorphic cotangent bundle. For a complex compact curve X of genus g, the degree of the canonical bundle is just the Euler characteristic

$$\deg\ \Omega^{n=1} = \chi(X) = 2(g-1).$$

Consequently, the only complex curves with trivial canonical class are those of genus $g = 1$, i.e. elliptic curves. All elliptic curves–indeed, all complex curves–are Kähler , equipped with both a symplectic structure ω and a complex structure τ. Hence, they are indeed Calabi-Yau manifolds.

All elliptic curves are quotients of Euclidean space with some symplectic structure $\omega =$ constant $dz \wedge d\bar{z}$, related to the volume form, by a lattice $\mathbb{Z} \oplus \tau\mathbb{Z}$, related to the complex structure. To fix notation, we label an elliptic curve E by symplectic form $\omega \in \mathbb{R}$ and complex structure $\tau \in \mathbb{C}$:

$$E_{\omega,\tau} = (\mathbb{C}, \omega)/\mathbb{Z} \oplus \tau\mathbb{Z}.$$

Since mirror symmetry relates Calabi-Yau 1-folds, given an elliptic curve $X = E_{\omega,(\tau)}$ (viewed as a symplectic manifold), mirror symmetry associates an elliptic curve $\tilde{X} = E_{(\tilde{\omega}),\tilde{\tau}}$ (viewed as a complex manifold). In this case, the topology of X and $\tilde{X}$ are the same, though this is not true in general. *What is the pairing?* The mirror manifold $\tilde{X}$ has complex structure $\tilde{\tau} = i\omega$ and extra symplectic structure $\tilde{\omega} = i\tau$.

$$\begin{array}{ccc} \text{Symplectic Side} & & \text{Holomorphic Side} \\ X = E_{\omega,(\tau)} & \leftrightarrow & \tilde{X} = E_{(i\tau),i\omega}. \end{array}$$

The parenthesis are part of the general principal that any any structure on X (indicated in parenthesis) has to correspond to extra structure on $\tilde{X}$. Consequently, though X is primarily viewed as a symplectic manifold, we might as well write down any additional structure and hope it's good for something.

Remark There's something odd in this pairing. It appears that the symplectic structure ω should be determined by a real constant, related to the overall rescaling of volume. But here, to make mirror symmetry work correctly, the symplectic form gets complexified. This may seem completely artificial, and classically, one might be skeptical. But the benefit is so beautiful! After doing a few exercises with this complexified symplectic structure, even skeptics would wonder why they didn't think of this earlier.

n = 2 : $X = K3$ or $X = T^4$

There are two compact complex surfaces which are Calabi-Yau, namely $X = K3$ and $X = T^4$, viewed as an abelian variety. In this case, it turns out that $\tilde{X}$ has the same topological type, i.e. the mirror of $X = K3$ is $\tilde{X} = K3$ and the mirror of $X = T^4$ is $\tilde{X} = T^4$.

Again, the symplectic and complex aspects of the Kähler structure of X and $\tilde{X}$ are exchanged. However, here both $K3$ and T^4 have even more structure than required: they both have hyperkähler structures. Rather than having a single symplectic structure and a single complex structure, they have infinitely many complex and symplectic structures. To fix notation, these complex structures are parameterized by a 2-sphere in the $3d$ space spanned by the three linearly independent complex structures I, J, K. The infinitely-many symplectic structures are parameterized by a 2-sphere in the $3d$-space spanned by the three associated linearly independent symplectic structures $\omega_I, \omega_J, \omega_K$. This extra hyperkähler structure on X corresponds to a hyperkähler structure on $\tilde{X}$, but with the roles of complex and symplectic structures exchanged.

n = 3 : Lots!

The world of compact Calabi-Yau 3-folds is very rich! There are many different manifolds–even topologically. One can easily concoct such examples by taking the orthogonal product of lower dimensional Calabi-Yau manifolds, e.g. $T^2 \times T^4$ and $T^2 \times K^3$. However, because there are so many Calabi-Yau 3-folds, we restrict our attention to the ones with finite fundamental group (i.e. some cyclic torsion, but no free factors), or – even better! – simply-connected ones.

Given X on the symplectic side, what can we say about $\tilde{X}$ on the holomorphic side? Generically, there is no additional structure that a Calabi-Yau 3-manifold has by virtue of its dimension. In low-dimensions, the condition that that the canonical bundle Ω^n of X was trivial was very restrictive; in higher-dimensions, the condition does not restrict much at all. Consequently, because there is no extra structure on X that must correspond to extra structure on $\tilde{X}$–effectively, restricting the pairing– the problem of pairing X and $\tilde{X}$ is more difficult.

There are three conjectural mirror symmetry correspondences for Calabi-Yau 3-folds that answer the questions of what manifolds have mirrors and what data on X and its mirror $\tilde{X}$ gets paired.

1.4 Mirror Symmetry Conjectures

1.4.1 Topological Mirror Symmetry

Since X is Calabi-Yau, hence Kähler , it has a Hodge structure. Note that the decomposition of cohomology

$$H^\bullet = \bigoplus_{p,q\geq 0} H^{p,q}(X)$$

comes from the complex structure on X, and the operator

$$\begin{array}{rcl} L : H^{p,q}(X) & \to & H^{p+1,q+1}(X) \\ \alpha & \mapsto & \alpha \wedge \omega \end{array}$$

comes from the symplectic structure on X. The Hodge decomposition gives numerical invariants of the manifold $h^{p,q}(X) = \dim H^{p,q}(X)$, arranged in the Hodge diamond of the manifold. The Hodge diamond of X is

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{2,2} & & 0 & \\ 1 & & h^{1,2} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & . \end{array}$$

On the border, the "1"s are from the Calabi-Yau structure of X, and indicate that there is a generator of $H^{p,q}(X)$; the "0"s are because $\pi_1(X)$ is a finite group. Because of Poincaré duality, $h^{2,2} = h^{1,1}$, and because of Serre duality, $h^{1,2} = h^{2,1}$. Consequently, the Hodge diamond of X has only two distinct numbers: $h^{1,1}$ and $h^{1,2}$.

What is the correspondence between the Hodge diamond of X and its mirror $\tilde{X}$? If X had a mirror $\tilde{X}$, the mirror would still be Calabi-Yau and and simply-connected, so the Hodge diamond would have the same border of 0's and 1's. As for the inside Hodge numbers, $H^{1,1}(X)$ controls the variation of symplectic structure while $H^{2,1}(X)$ controls the variation of the complex structure. Because mirror symmetry exchanges symplectic and complex structures, a variation of X's symplectic structure corresponds to to variation of $\tilde{X}$'s complex structure, and vise versa. Consequently, $h^{2,1}(\tilde{X}) = h^{1,1}(X)$ and $h^{1,1}(\tilde{X}) = h^{2,1}(X)$.

The **topological mirror symmetry conjecture** states:

Conjecture 1.1 *Let X be a Calabi-Yau 3-manifold X, having Hodge diamond*

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{1,2} & & h^{1,2} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & . \end{array}$$

Then there is a mirror Calabi-Yau 3-manifold $\tilde{X}$ whose Hodge diamond is rotated

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,2} & & 0 & \\ 1 & & h^{1,1} & & h^{1,1} & & 1 \\ & 0 & & h^{1,2} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & . \end{array}$$

1.4.2 Enumerative Mirror Symmetry

The enumerative mirror symmetry conjecture is an enhancement, in some sense, of the topological mirror symmetry conjecture. For topological mirror symmetry, a finite set of data on (X,ω) matches up with a finite set of data on $(\tilde{X},\tilde{\omega})$–namely, their respective Hodge diamonds. For enumerative mirror symmetry, infinite sets of data are related.

Corresponding to (X,ω) on the symplectic side there is $(\tilde{X},I)$ on the holomorphic side so that the Gromov-Witten enumerative invariants of (X,ω), normally computed on the symplectic side by counting stable holomorphic maps, packaged in some way, can instead be computed on the holomorphic side by certain periodic integrals $\int_\Gamma \Omega$ where $\Omega^3 = (\mathcal{T}^*X)^3$ is 3-form of $\tilde{X}$ and Γ are various cycles in $\tilde{X}$. From the earlier discussion on Gromov-Witten invariants, these cycles have something to do with where the Riemann surface in the σ-model lands in the target space $\tilde{X}$.

1.4.3 Homological Mirror Symmetry

The third version is homological mirror symmetry, mainly pioneered by Kontsevich (1994). This is the version we'll mainly focus on, since this is the version that branes appear in.

Homological mirror symmetry is an equivalence between the sets of "good objects" on the symplectic side (X,ω) and on the holomorphic side $(\tilde{X},I)$. What are the "good objects" on (X,ω)? These are the objects that are natural from the viewpoint of symplectic geometry. This includes Lagrangian submanifolds of X, which are mid-dimensional isotropic submanifolds (i.e. $L\subset X$ with $\dim_{\mathbb{R}} L = \dim_{\mathbb{C}} X$ such that $\omega|_L = 0$), as well as mid-dimensional coisotropic submanifolds. These natural symplectic objects are packaged together into the category $\mathrm{Fuk}(X,\omega)$, the **Fukaya category** of (X,ω).

On the holomorphic side, the natural objects include all holomorphic bundles–or more generally, sheaves– on the mirror manifold $(\tilde{X},I)$. These natural holomorphic objects are packed together into $D^\flat\mathrm{Coh}(\tilde{X},I)$, the **derived category of coherent sheaves**.

The **homological mirror symmetry conjecture** states:

Conjecture 1.2 *(Kontsevich) For any Calabi-Yau 3-manifold X, there is a mirror Calabi-Yau 3-manifold $\tilde{X}$ such that*

$$\mathrm{Fuk}(X,\omega) \cong D^\flat\mathrm{Coh}(\tilde{X},I).$$

(This equivalence is not functorial, though it should respect morphisms and possibly even some higher morphisms.)

What is the relation between these three different conjectures in mirror symmetry? The topological mirror symmetry conjecture involves the matching of finite sets of data on X and $\tilde{X}$. The enumerative mirror symmetry conjecture involves the pairing of the more restrictive infinite sets of the data on X and $\tilde{X}$. The homological mirror symmetry conjecture upgrades this pairing to categorical data.

In this sense, enumerative mirror symmetry is viewed as an enhancement of topological mirror symmetry. If topological mirror symmetry is false, then there's no chance that enumerative mirror symmetry is true since the dimension of the space of 3-cycles Γ in $\tilde{X}$, appearing in the periodic integrals, would be wrong. Likewise, homological mirror symmetry is viewed as an enhancement of both. This doesn't mean that the statement of homological mirror symmetry implies enumerative mirror symmetry as a corollary, or that enumerative mirror symmetry implies topological mirror symmetry as a corollary. The relation between the statements is complicated, and deriving one form of mirror symmetry from another is an open challenging problem that, recently, and Kevin Costello and Mohammed Abouzaid have been working on.

> **Note:** Each of these versions of mirror symmetry apply more generally than compact Calabi-Yau manifolds. Denis Auroux knows this story and what the mirrors of more general manifolds should be. Also, note that so far we've only been talking about compact Calabi-Yau manifolds. In the non-compact case, the moduli space of Higgs bundles is the pinnacle of the mirror symmetry story.

1.5 Branes

Physics use the words **A-model** and **B-model**. For our purposes, the A-model is the symplectic geometry of (X, ω) while the B-model is the complex geometry of $(\tilde{X}, I)$.

The word **brane** is completely synonymous with "object in ..." one of our categories. An A-brane is an object in $\text{Fuk}(X, \omega)$ and a B-brane is an object in $D^b\text{Coh}(\tilde{X}, I)$. There is a little more to "brane" than the cold phrase "object in a category" that we'll try to explain later. As Edward Frenkel said, "You can't love an object in a category, but you can love a brane."

We come back to branes in the third section, and give a more helpful definition. However, in the next section we shift gears a bit, and define Higgs bundles. These will be our primary example for mirror symmetry in the third section, because it's so rich!

2 The Hitchin Moduli Space

2.1 Definition

The Hitchin moduli space $\mathcal{M}_H(G, \Sigma)$ is labeled by a choice of compact Lie group G and Riemann surface Σ. The subscript H ambiguously stands for either "Higgs" or "Hitchin." The Hitchin moduli space can be defined in a number of ways. We'll use a slightly unusual formulation, in which things are phrased in unitary language rather than the standard holomorphic language, because the unitary language is suited for illustrating a particular isomorphism between solutions of Hitchin's equations and flat $G_{\mathbb{C}}$-connections.

Given Σ and G, let E be a hermitian vector bundle over Σ. Let A be unitary connection on E and ϕ be an adjoint $ad(E)$-valued 1-form on Σ. The Hitchin moduli space consists of the following pairs, moduli equivalence:

$$\mathcal{M}_H(G, \Sigma) = \left\{ (A, \phi) \;\middle|\; \substack{F_A - \phi\wedge\phi = 0, \\ d_A\phi = 0, \\ d_A^*\phi = 0} \right\} / \mathcal{G} \tag{5}$$

where $\mathcal{G}$ denotes the sets of G-gauge transformations acting by conjugation. The Hodge star comes from the hermitian metric on the bundle E.

Note that everything here is real. The real Lie algebra $\mathfrak{g}$– *not* the complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$ appears in the equation

$$F_A - \phi \wedge \phi \in \Omega^2(\Sigma, \text{End}(\mathfrak{g})),$$

Similarly, all other objects are also real:

$$\begin{array}{rcl} d_A\phi & \in & \Omega^2(\Sigma, \text{End}(\mathfrak{g})) \\ d_A^*\phi & \in & \Omega^0(\Sigma, \text{End}(\mathfrak{g})). \end{array}$$

If we complexify $\mathfrak{g}$, the curvature of the $G_{\mathbb{C}}$-connection $\mathcal{A} = A + i\phi$ is

$$\mathcal{F}_{\mathcal{A}} = (F_A - \phi \wedge \phi) + i(d_A\phi) = 0. \tag{6}$$

Observe that the first two equations defining the Hitchin moduli space can be repackaged as the real and imaginary parts of $\mathcal{F}_{\mathcal{A}} = 0$. The moduli space of flat $G_{\mathbb{C}}$-connections is

$$\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, \Sigma) = \{\mathcal{A} : \mathcal{F}_{\mathcal{A}} = 0\}/\mathcal{G}_{\mathbb{C}}$$

where $\mathcal{G}_{\mathbb{C}}$ is group of complex $G_{\mathbb{C}}$-gauge transformations acting by conjugation. We have the following theorem relating these two moduli spaces:

Theorem 2.1 $\mathcal{M}_H(G, \Sigma) \cong \mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, \Sigma)$

Note that in the $\mathcal{M}_H(G, \Sigma)$, we mod out by real gauge transformations, $\mathcal{G}$; in $\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, \Sigma)$, we are missing the additional equation $d_A^*\phi = 0$, but we mod out by the larger space of complex gauge transformations, $\mathcal{G}_{\mathbb{C}}$. This third equation plays the roll of a stability condition.

2.2 Properties

What structure do these moduli spaces have? $\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, \Sigma)$ has a complex structure, J, coming from the complexified structure of the gauge group, and a nice (complexified) symplectic structure ω which can be decomposed into the respective real and imaginary parts $\omega = \omega_I + i\omega_K$. Note that this data, $\{J, \omega_I, \omega_K\}$ is already half of a hyperkähler structure. The Hitchin moduli space inherits this structure from the above isomorphism. Indeed,

Theorem 2.2 *$\mathcal{M}_H(G, \Sigma)$ is hyperkähler .*

In this perspective, one can think of all three equations as moment map equations for the hyperkähler quotient on the infinite dimension space of pairs (A, ϕ). This is Hitchin's view. In addition, it turns out that the canonical class is trivial and consequently

Corollary 2.3 *$\mathcal{M}_H(G, \Sigma)$ is a non-compact Calabi-Yau manifold.*

The dimension of this moduli space can be computed easily computed from the flat connection aside. This dimension depends on the group and the topology of the Riemann surface.

2.3 Mirror Symmetry

Finally, since $X = \mathcal{M}_H(G, \Sigma)$ is already Calabi-Yau, it is natural to ask "Does it have a mirror? If so, what is it?" The answer to this question is "Yes!" The mirror manifold is also Calabi-Yau– but even more: it is a also a Hitchin moduli space, but for a different group. Mirror symmetry gives the following pairing:

$$X = \mathcal{M}_H(G, \Sigma) \leftrightarrow \tilde{X} = \mathcal{M}_H({}^L G, \Sigma)$$

where ${}^L G$ denotes the Langlands dual group.

Remark In number theory, while studying automorphic forms and Galois representations, Langlands noticed this nice relation between two problems: studying automorphic forms of one group and studying Galois representations of a completely different group. These days, this relation is called "Langlands Program", and has various ramifications in number theory, in geometry, and also has a geometric counterpart called the Geometric Langlands Correspondence. This Langlands Program involves pairing compact Lie groups. A few examples, Langlands duality preserves Cartan A,D,E types but changes topology, e.g. for $G = SU(2)$, ${}^L G = SO(3) = SU(2)/(\mathbb{Z}_2)$, and for $G = E_6$, ${}^L G = E_6/(\mathbb{Z}_3)$. Outside of A, D, E types, more interesting things happen. For example Langlands duality interchanges Cartan types B and C, exchanging symplectic and orthogonal groups.

So mirror symmetry also involves Langlands duality!

At what level do we know this mirror symmetry correspondence for the Hitchin moduli space? Do we know it at the topological level? At the enumerative level? At the homological level? At the topological level, this is a theorem due to Thaddeus and Hausel. But we don't know about the enumerative and homological levels.

In the next section, we return to branes, the objects in the categories appearing in the homological mirror symmetry correspondence. Our richest example is branes in the Hitchin moduli space, and we will give many examples.

3 Branes from the math perspective

3.1 Definition of "Brane"

Previously, we gave the following definition of the word "brane:"

Definition (Actual) A **brane** is an object of one of the following categories: On the A-side (i.e. symplectic side) of mirror symmetry, an A-brane is an object in the Fukaya category, $\mathrm{Fuk}(X)$.
On the B-side (i.e. complex side), a B-brane is an object in the derived category of coherent sheaves, $D^b\mathrm{Coh}(\tilde{X})$.

An alternate approximate definition of branes that is more useful when working with these things is that

Definition $'$(Approximate) A **brane** is a submanifold (of X or $\tilde{X}$, respectively)

How does this definition relate to the previous definition? Given a category, it is natural to ask what the building blocks of the category are. (The correct terminology for this is the "set of simple objects.") Objects in the Fukaya category $\mathrm{Fuk}(X)$ are built from Lagrangian submanifolds in the symplectic manifolds, and the Fukaya category is the category built to be a home for all such Lagrangian submanifolds. Though actual objects in $\mathrm{Fuk}(X)$ are often *complexes* of Lagrangian submanifolds, no one talks about such complexes; rather, they talk about single Lagrangian submanifolds, because if you understand the basic building block well-enough, there's no problem extending the story to composites of those building blocks.

On the B-side, the derived category of coherent sheaves is built from complexes of bundles or sheaves. How does one associate submanifolds in $\tilde{X}$ to building blocks of $D^b\mathrm{Coh}(\tilde{X})$? Given a sheaf, you can use the Grothendieck resolution of a sheaf to decompose it and represent it as a complex of bundles– and each bundle is supported on some subvariety. Equivalently, you can choose your basis of the derived category to be structure sheaves of holomorphic subvarieties or ideal sheaves. Each such sheaf, whether a structure sheaf or ideal sheaf, is supported on a subvariety. Consequently, a good first approximation to an object in the derived category, i.e. a B-brane, is a composition of holomorphic submanifold of $\tilde{X}$. Again, we'll just talk about a single holomorphic submanifold.

Now, the previous definition of a brane as a submanifold wasn't quite right. There is some extra data beyond a bare submanifold. The following definition, though imprecise, gives a broad summary of the data of a simple object in either $\mathrm{Fuk}(X)$ or $D^b\mathrm{Coh}(\tilde{X})$.

Definition $'$ A **brane** is a particular *submanifold* L (of X or $\tilde{X}$ respectively), a *vector bundle* $E \to L$ over it, with a *connection* A that satisfies certain conditions.

This data is ordered from important to less important. It is essential to remember the submanifold. In many cases the vector bundle E is fairly irrelevant and can be omitted. The connection is even less important, in general. The conditions on the connection depend on the side (A or B) and the submanifold. The following two sections on "A-branes" and "B-branes" will explain in more details the conditions on the submanifold, bundle, and connection in the definition above.

Note: This connection A on $E \to L$ is not to be confused with the connection appearing in Hitchin's equations the previous section.

3.2 A-branes

Historically, Fukaya introduced the Fukaya category as the complex of all Lagrangian submanifolds. However, people later realized that it was incomplete, i.e. there are B-branes in the derived category of coherent sheaves that didn't correspond to complexes of Lagrangian submanifolds via mirror symmetry. Since then, the Fukaya category has been *redefined* to be the category that makes mirror symmetry work. (However, both Fukaya's original definition and this new definition are used today. The context tells you the convention.) Describing the objects and morphisms in the Fukaya category is still a work in progress. We will discuss the objects, but not the morphisms.

There are two types of objects, i.e. A-branes, in $\mathrm{Fuk}(X)$.

(1) Generic A-branes: Lagrangian submanifolds
A generic A-brane is a Lagrangian submanifold $L \subseteq (X, \omega)$ (a mid-dimensional submanifold such that $\omega|_L = 0$), together with a unitary bundle $E \to L$ and flat connection A on E.

To illustrate the point that the submanifold is much more essential than the bundle with connection on it, note that if L is simply-connected, there are no interesting flat connections on a bundle.

(2) Special A-branes: Coisotropic submanifolds
This second type of A-brane is much more mysterious and is still a work in progress. Coisotropic submanifolds do not exist for "generic" (X, ω)– only for particular manifolds and particular symplectic structures. Often, if the manifold (X, ω) has lots of structure, one may find such additional objects. For example, $\mathbb{CP}^n$ has no coisotropic submanifolds. However, coisotropic submanifolds show up all the time in the structure-rich hyperkähler manifolds.

What are these objects? The submanifold L is no longer half-dimensional. Rather, the dimension of L satisfies

$$\dim_{\mathbb{R}} X \geq \dim_{\mathbb{R}} L > \frac{1}{2} \dim_{\mathbb{R}} X,$$

and the support of the objects in X goes up to the dimension of X itself, and includes $L = X$. The bundle and the connection are no longer unimportant. Rather, as $\dim L$ increases, the bundle becomes more and more important. The Fukaya category is still a work in progress. Currently, there is no global framework for describing the conditions imposed on the bundle and the connection. These are done in a case-by-case basis. Consequently, here we only describe the conditions on the bundle and connection in the extreme case where $L = X$. When $L = X$, $E \to X$ is a unitary bundle with connection A satisfying the non-linear equation $(F_A\omega^{-1})^2 = -\mathbb{1}$.

Coisotropic submanifolds will be appear in the Hitchin moduli space, our main example, and are part of the quantization story. Such coisotropic submanifolds appear because of the hyperkähler structure on the Hitchin moduli space. One step in quantizing a symplectic manifold (X, ω_I) involves finding a line bundle $\mathcal{L} \to X$ and a connection A whose curvature is the symplectic 2-form, i.e. $F_A = \omega_I$. If one then switches perspectives and views X as a symplectic manifold with different symplectic form $\omega = \omega_K$, then the connection A satisfies the above non-linear equation:

$$\begin{aligned}(F_A\omega^{-1})^2 &= (\omega_I\omega_K^{-1})^2 \\ &= (J)^2 \\ &= -\mathbb{1}.\end{aligned}$$

3.3 B-branes

Mirror symmetry gives a correspondence between A-branes and B-branes. As you'd hope, the special coisotropic A-branes correspond to special B-branes; the generic type of A-branes, namely Lagrangian submanifolds, correspond to generic B-branes. We briefly describe these two classes of objects in the derived category, $D^\flat\mathrm{Coh}(\tilde{X})$.

(1) Generic B-branes: Holomorphic submanifolds

The generic simple objects in $D^\flat\mathrm{Coh}(\tilde{X})$ are sheaves, in $\tilde{X}$. We can associate a holomorphic submanifold to a sheaf by taking the support of the sheaf. Though holomorphic submanifolds, $\tilde{L} \subseteq (\tilde{X}, I)$, are only an approximation to the story, they are a *good* approximation and they capture essential information.

(2) "Special B-branes"

"Special B-branes" exist for only special manifolds $\tilde{X}$ or special complex structures. These special choices of complex structures are precisely where the derived category jumps and there are very special additional objects that we don't see for generic choice of complex structure. For example, if the complex manifold $(\tilde{X}, I)$ has some particular symmetry, such as a notion of complex multiplication, then these extra objects appear. For example, elliptic curves E_τ have a notion of complex multiplication when τ, the parameter of the complex structure satisfies

$$a\tau^2 + b\tau + c = 0$$

for $a, b, c \in \mathbb{Z}$ with $b^2 - 4ac < 0$. It is at these special complex structures that the derived category jumps and extra objects appear. If you perturb the complex structure τ, you destroy the integrality. Note that on the A-side, this integrality condition appeared implicitly: the non-linear equation $(F_A\omega^{-1})^2 = -\mathbb{1}$ imposed an integrality condition on the curvature: $[F_A] \in H^2(X, \mathbb{Z})$.

3.4 A-branes and B-branes in moduli space of Higgs bundles

Our main example for mirror symmetry with be the Hitchin moduli space. Mirror symmetry pairs $X = \mathcal{M}_H(G, \Sigma)$ and $\tilde{X} = \mathcal{M}_H({}^LG, \Sigma)$. This example is particularly rich, and as such, is a good laboratory for exploring A-branes and B-branes.

Let's actually consider a baby version of Hitchin moduli space, to make things very concrete. More explicitly, if $G = U(1)$, then $X = T^*\mathrm{Jac}(\Sigma) = \mathbb{R}^n \times \mathrm{Jac}(\Sigma)$ since the bundle is trivial. Even more concretely, if $\Sigma = T^2$, then $X = \mathbb{R}^2 \times T^2$. Let x_0, x_1 be coordinates on $\mathbb{R}^2$ and x_2, x_3 coordinates on T^2. The symplectic forms are

$$\begin{aligned}\omega_I &= dx^0 \wedge dx^1 + dx^2 \wedge dx^3 \\ \omega_J &= dx^0 \wedge dx^2 - dx^1 \wedge dx^3 \\ \omega_K &= dx^0 \wedge dx^3 + dx^1 \wedge dx^2\end{aligned}$$

The complex structures I, J, K are determinedby these symplectic forms. With respect to the holomorphic coordinates

$$\begin{aligned} z_I &= x_0 + ix_1 & w_I &= x_2 + ix_3 \\ z_J &= x_0 + ix_2 & w_J &= x_3 + ix_1 \\ z_K &= x_0 + ix_3 & w_K &= x_1 + ix_2, \end{aligned}$$

I, J, K just act by multiplication by $\sqrt{-1}$, e.g.

$$\begin{aligned} I \cdot dz_I &= \sqrt{-1}dz_I, \\ I \cdot dw_I &= \sqrt{-1}dw_I, \quad \text{etc.} \end{aligned}$$

What do A-branes look like for (X, ω_J)*?* There are plenty!

Lagrangian submanifolds of (X, ω_J) include:

- $L = \mathbb{R}^2_{(x_0,x_1)} \times \text{pt} \in T^2_{x_2,x_3}$. This is trivial example of the Hitchin fibration, and L is copy of the Hitchin base. One can check that $\omega_| L$ vanishes.
- $L = \text{pt} \times T^2$. This is special case of choosing particular fiber.

Because $\mathcal{M}_H(U(1), T^2)$ is hyperkähler , we should check whether the above submanifolds L are holomorphic or Lagrangian with respect to the other structures. In both of the above examples, L is holomorphic with respect to the complex structure I and Lagrangian with respect to both ω_J and ω_K.

What should we call such a submanifold? There are lots of true statements, depending on which structure we are viewing it from. It's a B-brane in (X, I), an A-brane in (X, ω_J) and an A-brane in (X, ω_K).

This illustrates the point that a brane is more than just an object in a category. To describe a brane as an object in a category we have to pick a single viewpoint (e.g. "Is L a Lagrangian submanifold in X equipped with choice of symplectic structure ω_J?"). But one viewpoint is not enough! Consequently, we call the above submanifolds (B, A, A)-branes, keeping track of three different viewpoints. The first slot refers to the Kähler structure (X, I, ω_I), the second to (X, J, ω_J) and the third to (X, K, ω_K).

What other types of branes exist?

Examples of (A, B, A) branes in $\mathbb{R}^2 \times T^2$ include the cylinders described by $\{x_1 = \text{pt}\} \times \{x_3 = \text{pt}\}$ and $\{x_0 = \text{pt}\} \times \{x_2 = \text{pt}\}$. Examples of (A, A, B) branes, similarly, include the cylinders $\{x_1 = \text{pt}\} \times \{x_2 = \text{pt}\}$ and $\{x_0 = \text{pt}\} \times \{x_3 = \text{pt}\}$.

However, it is not possible to produce an example of an (A, B, B)-brane or a (B, A, B)-brane or (B, B, A)-brane. If a Lagrangian submanifold is holomorphic with respect to two complex structure–a very restrictive condition–then it must be holomorphic with respect to the third complex structure.

Indeed, (B, B, B)-branes do exist! (B, B, B) branes are called "hyper-holomorphic," as well as "tri-analytic" because they are holomorphic with respect to all three structures. This is a very restrictive condition. It is hard to make a curved thing, like a submanifold, holomorphic with respect to two, let alone three, complex structures. Consequently, (B, B, B) branes are special and rare. (B, B, B)-branes do not appear in general hyperkähler manifolds. However, (B, B, B) appear in the Hitchin moduli space because of the non-compactness of the hyperkähler space.

Trivial examples of (B, B, B)-branes –though not a mid-dimensional submanifolds– include the whole manifold X and the point with skyscraper sheaf. Non-trivial examples are much more difficult to produce.

These special (B, B, B)-branes are dual to (B, A, A)-branes. In mirror symmetry, nothing gets lost, and nothing is gained–though it may change. Consequently, building more (B, A, A)-branes is a concrete way to understand the more mysterious (B, B, B)-branes. The following are also (B, A, A)-branes:

- Take $z = z_I = x_0 + ix_1$ and $w = w_I = x_2 + ix_3$, holomorphic coordinates with respect to the complex structure I. For any holomorphic function f, the variety defined by $\{f(z, w) = 0\}$ is clearly holomorphic in I. It is easy to check that it is a Lagrangian submanifold with respect to ω_J and ω_K.

- For a more interesting example that generalizes to any Hitchin moduli space, take the whole space $L = X = \mathcal{M}_H(G,\Sigma)$ as the submanifold. Let $\mathcal{L} \to X$ be a line bundle with Chern class $c_1(\mathcal{L}) = \omega_I$. (Note that this means that $\mathcal{L} \to X$ is a holomorphic object with respect to I.) Let A be a connection on $\mathcal{L}$ such that $F_A = \omega_I$. This is a B-brane with respect to the symplectic forms ω_J and ω_K because the non-linear equation $(F_A\omega^{-1})^2 = -\mathbb{1}$ is satisfied, i.e. both $(\omega_I\omega_J^{-1})^2 = -\mathbb{1}$ and $(\omega_I\omega_K^{-1})^2 = -\mathbb{1}$.

The study of branes in the Hitchin moduli space is a very rich subject! The above example illustrates the point that if you keep track of all the structure, you get a lot more leverage and fun phenomena.

4 Quantization

Since the title of the talk was officially "Quantization via Mirror Symmetry," we'll end by saying a few things about quantization. Quantization seeks to associate a Hilbert space to a symplectic manifold. In the context of branes, to physicists, this Hilbert space $\mathcal{H}$ is the space of open string states between two A-branes, $\mathcal{B}'$ and $\mathcal{B}_{cc}$, where $\mathcal{B}'$ is an ordinary Lagrangian A-brane, and $\mathcal{B}_{cc}$ is some particular type of coisotropic A-brane. To mathematicians, this Hilbert space is the space of morphisms

$$\mathcal{H} = \mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}')$$

between the two objects in the Fukaya category.

For more of this rich story, see the notes "Quantization via Mirror Symmetry" (arXiv:1011.2218).

5 Suggested Exercises

1. Give an explicit description of the moduli space of Higgs bundles $\mathcal{M}_H(G,\Sigma)$ in each of the following examples:
   (a) $G = U(1)$ and $\Sigma =$ genus-g Riemann surface
   (b) $G = SU(2)$ and $\Sigma = T^2$
   (c) $G = SU(2)$ and $\Sigma = T^2\backslash\{p\}$ with a "first order pole" at p
   (d) $G = SU(2)$ and $\Sigma = S^2\backslash\{p_1,p_2,p_3,p_4\}$ with tame ramifications at p_1,p_2,p_3, and p_4.
   (e) $G = SU(2)$ and $\Sigma =$ genus-2 Riemann surface (without punctures)
2. In each of the above examples, propose a mirror manifold to $\mathcal{M}_H(G,\Sigma)$.
3. Pick a 3-manifold M^3 bound by a 2-torus $T^2 = \partial M^3$ and describe the corresponding (A,B,A)-brane $\mathcal{B}$ in the Hitchin moduli space in the exercise $(1b)$. The brane $\mathcal{B}$ is supported on $\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, M^3)$.
4. Do the same for a genus-2 Riemann surface Σ: pick your favorite genus-2 handlebody M^3 and describe the (A,B,A)-brane $\mathcal{B}$ supported on the moduli space of flat $SL(2,\mathbb{C})$ connections on the Hitchin moduli space in exercise (1e).
5. Combine the results of exercises (2), (3), and (4) to find the mirror branes $\tilde{\mathcal{B}}$ in each case.
6. In the context of exercise $(1a)$, find the mirror of the (B,A,A)-brane $\mathcal{B}$ supported on the the base (respectively, the fiber) of the Hitchin fibration. In each case, verify that the resulting brane is indeed of type (B,B,B), that is hyper-holomorphic.
7. In each case (b)– (e) of execise (1), describe the (B,A,A)-brane, $\mathcal{B}$ supported on a "real slice"–that is, supported on $\mathcal{M}_{\text{flat}}(G_{\mathbb{R}},\Sigma)$ where $G_{\mathbb{R}}$ is either $SU(2)$ or $SL(2,\mathbb{R})$.
8. Combine the results of exercises (2) and (7) to find the mirror (B,B,B) branes $\tilde{B}$ in each case.

References

[1] S. Gukov, Takagi Lecture: *Quantization via Mirror Symemtry*, Japan. J Math. **6** (2011) 65119, arXiv:1011.2218.