Hyperkähler metrics near semiflat limits

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Data:
- $\mathcal{B}$ complex manifold, $r = \dim_{\mathbb{C}} \mathcal{B}$
  $\cup$
  $\mathcal{B}'$
- exact sequence of lattices over $\mathcal{B}'$

$$0 \longrightarrow \underset{\text{(flavor lattice)}}{\Gamma_f} \longrightarrow \hat{\Gamma} \longrightarrow \underset{(\dim \Gamma = 2r)}{\Gamma} \longrightarrow 0$$

w/ antisymmetric pairing $\langle , \rangle : \hat{\Gamma} \times \hat{\Gamma} \longrightarrow \mathbb{Z}$ such that $\Gamma_f = \{\gamma \in \hat{\Gamma} : \langle \gamma, \gamma' \rangle = 0 \text{ for all } \gamma'\}$

- $\Omega : \hat{\Gamma} \longrightarrow \mathbb{Z}$ allowed to jump along $\operatorname{codim}_{\mathbb{R}} 1$ walls in $\mathcal{B}'$ satisfying wall-crossing formula and various other properties
  Notation: $\Omega(\gamma, u)$
- $Z : \hat{\Gamma} \longrightarrow \mathbb{C}$ "charges"
  Notation: $Z_\gamma(u)$
- $\theta_f : \Gamma_f \longrightarrow \mathbb{R}/2\pi\mathbb{Z}$

$\rightsquigarrow$ Integrable system $\Gamma \otimes_{\mathbb{Z}} \mathbb{R}/2\pi\mathbb{Z} = \mathcal{M}'$ with fiber coords θ_γ, $\gamma \in \Gamma$

$$\begin{array}{c} \mathcal{M}' \\ \downarrow \\ \mathcal{B}' \end{array}$$

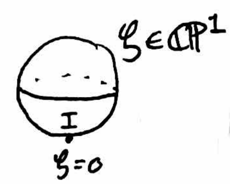
w/ natural singular hyperkähler metric known as semiflat metric

semiflat holomorphic Darboux coords

$$\mathcal{X}^{sf}_\gamma : \mathcal{M}' \times \mathbb{C}^\times \longrightarrow \mathbb{C}^\times$$
$$(u, \theta, \zeta) \longmapsto \exp\left[\pi R \frac{Z_\gamma}{\zeta} + i\theta_\gamma + \pi R \zeta \bar{Z}_\gamma\right]$$

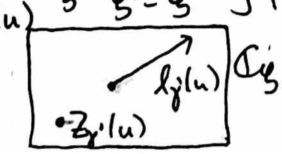
family of holomorphic symplectic forms

$$\varpi_R(\zeta) = \frac{1}{8} \langle d\log \mathcal{X}^{sf}(\zeta) \wedge d\log \mathcal{X}^{sf}(\zeta) \rangle$$
$$= \frac{1}{\zeta} \Omega_{I,R} + \omega_{I,R} + \zeta \overline{\Omega_{I,R}}$$



True hyperkähler metric from integral relations: $\mathcal{X}_\gamma : \mathcal{M}' \times \mathbb{C}^\times \longrightarrow \mathbb{C}^\times$ satisfy

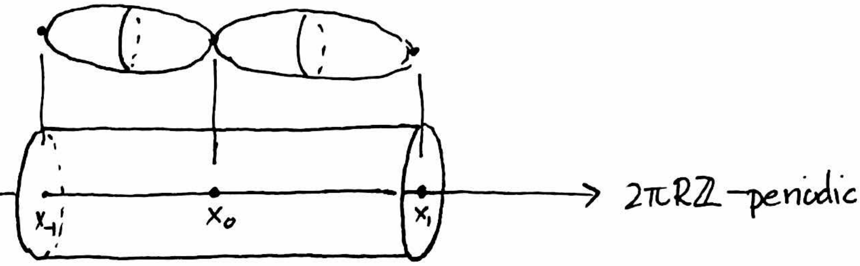
$$\mathcal{X}_\gamma(u,\theta,\zeta) = \mathcal{X}^{sf}_\gamma(u,\theta,\zeta) \exp\left[-\frac{1}{4\pi i} \sum_{\gamma'} \underbrace{\Omega(\gamma', u)\langle \gamma, \gamma' \rangle}_{\text{correction from } \mathcal{X}_{\gamma'} \text{ iff nonzero}} \int_{\ell_{\gamma'}(u)} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log\left(1 - \mathcal{X}_{\gamma'}(u,\theta,\zeta')\right)\right]$$



Simplest Example: Ooguri-Vafa

Gibbons-Hawking ansatz presentation:

$$\begin{array}{c} S^1_{\theta_m} \hookrightarrow \widetilde{M} \\ \downarrow \\ \mathbb{D}_u \times \mathbb{R}_{\tilde{\theta}_e} \end{array}$$



positive harmonic function on $\mathbb{D}_u \times \mathbb{R}_{\tilde{\theta}_e}$:

$$V = \frac{1}{4\pi\|x\|} + \sum_{n\in\mathbb{Z}-\{0\}} \left(\frac{1}{4\pi\|x-x_n\|} - \underbrace{\frac{1}{8\pi^2|n|}}_{\text{for convergence}} \right) + \underbrace{C}_{\text{for positivity}}$$

$\rightsquigarrow$ hyperkähler metric on $\widetilde{M}$

$$\tilde{g} = V\|dx\|^2 + V^{-1}\Theta^2 \qquad \Theta = d\theta_m + A \in \Omega^1(\widetilde{M}) \text{ satisfying } d\Theta = F = -2\pi * dV$$

Ooguri-Vafa is $\widetilde{M}/_{2\pi R\mathbb{Z}}$

GMN presentation:

Lattices: Γ_f trivial $\qquad \hat{\Gamma} = \Gamma = \mathrm{Span}(\gamma_m, \gamma_e) \qquad \langle \gamma_m, \gamma_e \rangle = 1$

Across cut:

$\gamma_e \longmapsto \gamma_e$
$\gamma_m \longmapsto \gamma_m + \gamma_e$

BPS Data: $\Omega(\gamma, u) = \begin{cases} 1 & \text{if } \gamma = \pm\gamma_e \\ 0 & \text{otherwise} \end{cases}$

Charges: $Z_{\gamma_e}(u) = u$

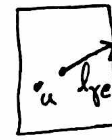
$$Z_{\gamma_m}(u) = \frac{u}{2\pi i}(\log u - 1)$$

Integrable system: $R = \frac{1}{\pi}$

$$\mathcal{X}^{sf}_{\gamma_e} = \exp\left[\zeta^{-1} Z_{\gamma_e}(u) + i\theta_{\gamma_e} + \zeta \bar{Z}_{\gamma_e}(u)\right]$$
$$\mathcal{X}^{sf}_{\gamma_m} = \exp\left[\zeta^{-1} Z_{\gamma_m}(u) + i\theta_{\gamma_m} + \zeta \bar{Z}_{\gamma_m}(u)\right]$$

True hyperkähler metric:

jump along



$\boxed{\gamma_e}$ $\Omega(\gamma', u)\langle \gamma_e, \gamma' \rangle = 0$ for all γ' $\Rightarrow \mathcal{X}_{\gamma_e} = \mathcal{X}^{sf}_{\gamma_e}$

$\boxed{\gamma_m}$ $\Omega(\gamma', u)\langle \gamma_m, \gamma' \rangle = \begin{cases} \pm 1 & \text{if } \gamma = \pm\gamma_e \\ 0 & \text{otherwise} \end{cases} \Rightarrow \mathcal{X}_{\gamma_m} = \mathcal{X}^{sf}_{\gamma_m} \exp\left[\frac{-1}{4\pi i}\sum_{\pm}(\pm 1)\int_{\ell_{\pm\gamma_e}} \frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\log(1-\mathcal{X}_{\gamma_e})\right]$

Note: See GMN 0807.4723 §4.3 for verification that recipe produces Ooguri-Vafa. Fundamental computation is

① $\int_\ell \frac{d\zeta'}{\zeta'}\frac{\mathcal{X}_e(\zeta')}{1-\mathcal{X}_e(\zeta')} = \sum_{n=1}^{\infty} 2e^{in\theta_m} K_0(2n|u|)$ ② $V = -\frac{1}{4\pi^2}\log\left|\frac{u}{\Lambda}\right| + \frac{1}{2\pi^2}\cos(n\theta_e)K_0(2n|u|)$

modified Bessel function of 1st kind