GAUGE THEORY, GRAVITATION, AND GEOMETRY

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ABSTRACT. The study of manifolds, and metrics and connections over them, has many profound links with modern theoretical physics. In particular, geometric invariants and deformation problems are closely connected to the way that particles and their interactions are described in gauge theory. Important connections also exist to theories of gravitation (including Einstein's theory of general relativity) and many others, including string theory.

In this course we will provide an introduction to the geometry of manifolds and vector bundles oriented towards discussing gauge theories. We'll highlight some famous and interesting gauge theories through concrete computation including: the Yang-Mills theories (which give rise to the standard model in particle physics), Chern-Simons theories (which have been used to compute knot invariants), Einstein's Field Equations for gravitation, and Kaluza-Klein type-theories (a class of unified field theories that unify gravitation and electromagnetism).

Prerequisites: Linear Algebra, multivariable calculus, ordinary differential equations, and a first course in abstract algebra (basic group theory). Some familiarity with differential geometry at the level of curves and surfaces would be extremely helpful.

Lecture 1: Overview of Maxwell's Equations (LF). [(Monday, July 1, 2019)]

- Maxwell's equations in terms of electric and magnetic fields **E**, **B**
- Wave-like nature of Maxwell's equations
- Maxwell's equations in terms of electric and magnetic potentials φ and **A**
- Introduction to gauge invariance in the context of Maxwell's equations

Lecture 2: Manifolds & Vector Fields (LF). [(Tuesday, July 2, 2019)]

- Introduction to manifolds, including classification of 2-manifolds.
- Vector field on manifolds.

Lecture 3: Differential 1-Forms & Exterior Derivative & Differential Forms (LF). [(Wednesday, July 3, 2019)]

- Differential 1-forms on manifolds
- Differential k-forms on manifolds
- The exterior derivative d on manifolds

Date: June 30, 2019.

Lecture 4: Metrics & Hodge Star (LF). [(Friday, July 5, 2019)]

- Introduction to metrics on manifolds
- The Hodge star operator $\star$
- The exterior derivative is gradient, divergence, and curl all-in-one.

Lecture 5: Maxwell's Equations Revisited (LF). [(Monday, July 8, 2019)]

- Rewrite Maxwell's equations compactly as $\mathrm{d}F = 0$ and $\star\mathrm{d}\star F = j$.
- Introduction to Lie groups $SL(n, \mathbb{R}), SO(n)$, etc.
- Symmetries of Maxwell's equations

Lecture 6: DeRham Theory in Electromagnetism & Hodge Theory (LF). [(Tuesday, July 9, 2019)]

- Aharanov-Bohm effect
- Introduction to Hodge theory

Lecture 7: Symmetries (LF). [(Wednesday, July 10, 2019)]

- Lagrangian formulation of mechanics
- Symmetries give rise to conserved quantities

Lecture 8: Vector Bundles (LA). [(Thursday, July 11, 2019)]

- Introduction to fiber and vector bundles
- Examples of bundle constructions
- Local trivializations
- Introduction to connections

Lecture 9: Connections and Yang-Mills Curvature (LA). [(Friday, July 12, 2019)]

- Continuation on connections
- Yang-Mills curvature
- The physical significance of Yang-Mills/Gauge theory (examples from modern particle physics)

Lecture 10: Yang-Mills Equations + Chern Classes (LA). [(Monday, July 15, 2019)]

- Yang-Mills Equations (and the Standard Model of particle physics)
- Invariants, Chern classes
- The Chern-Simons form

Lecture 11: Introduction to Tensors (LA). [(Tuesday, July 16, 2019)]

- The notion of a tensor
- Linking tensors and differential forms
- The Levi-Civita connection

Lecture 12: Tensors for General Relativity (LA). [(Wednesday, July 17, 2019)]

- General diffeomorphism invariance
- Notions of curvature
- Riemann/Ricci tensors

Lecture 13: Einstein's Equations (LA). [(Thursday, July 18, 2019)]

- Einstein's equations
- A scenic tour of General Relativity

Lecture 14: Modern outlook (LA). [(Friday, July 19, 2019)]

- An introduction to Kaluza-Klein theory
- An outlook on the role of the tools introduced in this course in modern physics and geometry.

Supplementary Material. We recommend John Baez & Javier P. Muniain's book *Gauge Fields, Knots and Gravity*.

Note: Since we aim to make this an interactive course, the schedule above may naturally flex/shift. So please take this schedule as a rough guide only.