GAUGE THEORY, GRAVITATION, AND GEOMETRY
PROBLEMS & SOLUTIONS

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ABSTRACT. The study of manifolds, and metrics and connections over them, has many profound links with modern theoretical physics. In particular, geometric invariants and deformation problems are closely connected to the way that particles and their interactions are described in gauge theory. Important connections also exist to theories of gravitation (including Einstein's theory of general relativity) and many others, including string theory.

In this course we will provide an introduction to the geometry of manifolds and vector bundles oriented towards discussing gauge theories. We'll highlight some famous and interesting gauge theories through concrete computation including: the Yang-Mills theories (which give rise to the standard model in particle physics), Chern-Simons theories (which have been used to compute knot invariants), Einstein's Field Equations for gravitation, and Kaluza-Klein type-theories (a class of unified field theories that unify gravitation and electromagnetism).

Prerequisites: Linear Algebra, multivariable calculus, ordinary differential equations, and a first course in abstract algebra (basic group theory). Some familiarity with differential geometry at the level of curves and surfaces would be extremely helpful.

1. LECTURE 1: MAXWELL'S EQUATIONS (LF)

[(Monday, July 1, 2019)]

1.1. Maxwell's equations.

- electric field $\mathbf{E} = \mathbf{E}(\mathbf{x}, t)$ for $\mathbf{x} \in \mathbb{R}^3, t \in \mathbb{R}$
- magnetic field $\mathbf{B} = \mathbf{B}(\mathbf{x}, t)$
- electric charge density $\rho = \rho(\mathbf{x}, t)$
- electric current density $\mathbf{j} = \mathbf{j}(\mathbf{x}, t)$

In units where the speed of light c is equal to 1 and $\epsilon = 1, \mu = 1$ **Maxwell's equations** are

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= \mathbf{0} \\ \nabla \cdot \mathbf{E} &= \rho \\ \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} &= \mathbf{j} \end{aligned} \tag{1.1}$$

Date: July 9, 2019.

1.2. **Wave-like nature of solutions of Maxwell's equations.** It follows that $\mathbf{E}$ and $\mathbf{B}$ solve

$$\begin{aligned} \nabla^2\mathbf{E} - \frac{\partial^2\mathbf{E}}{\partial t^2} &= \nabla\rho + \frac{\partial\mathbf{j}}{\partial t} \\ \nabla^2\mathbf{B} - \frac{\partial^2\mathbf{B}}{\partial t^2} &= -\nabla\times\mathbf{j}. \end{aligned}$$

1.3. **Gauge invariance in Maxwell's equations.**

- electric potential $\varphi = \varphi(\mathbf{x}, t)$
- magnetic potential $\mathbf{A} = \mathbf{A}(\mathbf{A}, t)$

If

$$\mathbf{B} = \nabla\times\mathbf{A}, \qquad \mathbf{E} = -\nabla\varphi - \frac{\partial}{\partial t}\mathbf{A}, \tag{1.2}$$

then Maxwell's equations become the following pair of equations for $(\varphi, \mathbf{A})$:

$$\begin{aligned} \nabla^2\varphi + \frac{\partial}{\partial t}(\nabla\cdot\mathbf{A}) &= -\rho \qquad (1.3)\\ \left(\nabla^2\mathbf{A} - \frac{\partial^2}{\partial t^2}\mathbf{A}\right) - \nabla\left(\nabla\cdot\mathbf{A} + \frac{\partial\varphi}{\partial t}\right) &= -\mathbf{j}. \end{aligned}$$

(The laws $\nabla\cdot\mathbf{B} = 0$ and $\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$ are trivially satisfied.)

The potentials $(\varphi, \mathbf{A})$ uniquely determine the fields $(\mathbf{B}, \mathbf{E})$, but the fields do not uniquely determine the potentials, e.g. given a real-valued function Γ on spacetime $\mathbb{R}^3_{\mathbf{x}}\times\mathbb{R}_y$, define

$$\begin{aligned} \varphi' &= \varphi + \frac{\partial}{\partial t}\Gamma \qquad (1.4)\\ \mathbf{A}' &= \mathbf{A} - \nabla\Gamma. \end{aligned}$$

Then the fields $(\mathbf{B}', \mathbf{E}')$ associated to the potentials $(\varphi', \mathbf{A}')$ are equal to the fields $(\mathbf{B}, \mathbf{E})$ associated to the potentials $(\varphi, \mathbf{A})$. We call a change of potentials that does not change the fields a **gauge transformation**.

Daily Exercises.

Exercise 1.1. [Straightforward] Derive Maxwell's equations in (1.3) from the expression of Maxwell's equations in (1.1).

Solution: Maxwell's equations are

$$\begin{aligned} \nabla\cdot\mathbf{B} &= 0 \qquad (1.5)\\ \nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} &= \mathbf{0} \\ \nabla\cdot\mathbf{E} &= \rho \\ \nabla\times\mathbf{B} - \frac{\partial\mathbf{E}}{\partial t} &= \mathbf{j}. \end{aligned}$$

For the first equation, we compute

$$\begin{aligned} 0 &= \nabla \cdot \mathbf{B} \\ &= \nabla \cdot \nabla \times \mathbf{A}. \end{aligned}$$

This is trivially satisfies since the divergence of the curl of a vector field is zero.

For the second equation, we compute

$$\begin{aligned} 0 &= \nabla \times (-\nabla \varphi - \partial_t \mathbf{A}) + \frac{\partial}{\partial t}(\nabla \times \mathbf{A}) \\ &= -\nabla \times \nabla \varphi. \end{aligned}$$

This is trivially satisfied since the curl of the gradient of a function is zero.

For the third equation, we compute

$$\begin{aligned} \rho &= \nabla \cdot \mathbf{E} \\ &= \nabla \cdot \left(-\nabla \varphi - \frac{\partial}{\partial t} \mathbf{A} \right) \\ &= -\left(\nabla^2 \varphi + \frac{\partial}{\partial t} \mathbf{A} \right). \end{aligned}$$

This gives the first equation in (1.3).

For the last equation, we compute

$$\begin{aligned} \mathbf{j} &= \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} \\ &= \nabla \times (\nabla \times \mathbf{A}) - \frac{\partial}{\partial t}\left(-\nabla \varphi - \frac{\partial}{\partial t} \mathbf{A} \right) \\ &= -\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) + \frac{\partial^2}{\partial t^2}\mathbf{A} + \frac{\partial}{\partial t}\nabla \varphi \\ &= -\left(\left(\nabla^2 \mathbf{A} - \frac{\partial^2}{\partial t^2}\mathbf{A} \right) - \nabla \left(\nabla \cdot \mathbf{A} + \frac{\partial \varphi}{\partial t} \right) \right). \end{aligned}$$

This gives the second equation in (1.3).

Exercise 1.2. The electric and magnetic potentials φ and $\mathbf{A}$ are said to be "in Lorenz gauge" if

$$\nabla \cdot \mathbf{A} + \frac{\partial \varphi}{\partial t} = 0. \tag{1.6}$$

(a) Maxwell's equations drastically simplify in Lorenz gauge. Find these simpler equations.

(b) Determine electric and magnetic potentials $\varphi, \mathbf{A}$ for the plane wave

$$\mathbf{E} = \begin{pmatrix} 0 \\ \phi_2(x-t) + \psi_2(x+t) \\ \phi_3(x-t) + \psi_3(x+t) \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 0 \\ -\phi_3(x-t) + \psi_3(x+t) \\ \phi_2(x-t) - \psi_2(x+t) \end{pmatrix}. \tag{1.7}$$

such that $\varphi, \mathbf{A}$ are in Lorenz gauge, i.e.

$$\nabla \cdot \mathbf{A} + \frac{\partial \varphi}{\partial t} = 0. \tag{1.8}$$

[Hint: In this gauge $\mathbf{A}$ and φ will only depend on x and t.]

(c) Show that your solution to (b) is also in Coulomb gauge, i.e. $\nabla \cdot \mathbf{A} = 0$.

Solution: (a) Maxwell's equations simplify to

$$\nabla^2 \varphi - \frac{\partial^2}{\partial t^2}\varphi = -\rho$$
$$\nabla^2 \mathbf{A} - \frac{\partial^2}{\partial t^2}\mathbf{A} = -\mathbf{j}.$$

(b) Assuming that $\mathbf{A}$ depends only on x and t, we see that

$$\mathbf{A} = \begin{pmatrix} \Phi_1(x-t) - \Psi_1(x-t) \\ \Phi_2(x-t) - \Psi_2(x-t) \\ \Phi_3(x-t) - \Psi_3(x+t). \end{pmatrix} \tag{1.9}$$

We then compute

$$\begin{aligned} \mathbf{B} &= \nabla \times \mathbf{A} \\ &= \begin{pmatrix} 0 \\ -\partial_x \left(\Phi_3(x-t) - \Psi_3(x+t)\right) \\ \partial_x \left(\Phi_2(x-t) - \Psi_2(x+t)\right) \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ -\left(\Phi_3'(x-t) - \Psi_3'(x+t)\right) \\ \left(\Phi_2'(x-t) - \Psi_2'(x+t)\right) \end{pmatrix} \end{aligned}$$

We then match our components and find that

$$\Psi_3' = \psi_3, \qquad \Phi_3' = \phi_3, \qquad \Psi_2' = \psi_2, \qquad \Phi_2' = \phi_2. \tag{1.10}$$

We then see that $\frac{\partial \mathbf{A}}{\partial t} = \mathbf{E}$, hence the electric potential satisfies $\varphi = 0$. This trivially satisfies the Lorenz gauge condition since the first entry of $\mathbf{A}$ is zero and $\varphi = 0$.

Note: There is actually a bit of freedom left to change the first component of $\mathbf{A}$.

(c) It's obviously that this particular solution is in Coulomb gauge again since the first entry of $\mathbf{A}$ is zero.

Exercise 1.3 (Conservation)**.** [Straightforward] Suppose that U and $\mathbf{V}$ are a scalar field and a vector field satisfying the relationship

$$\frac{\partial U}{\partial t} + \nabla \cdot \mathbf{V} = 0 \tag{1.11}$$

show that this equation can be interpreted as *conservation* of U, in the sense that it implies that the rate of decrease of the total amount of U in a region R equals the rate of flux of $\mathbf{V}$ out of R.

From Maxwell's equations, derive that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0. \tag{1.12}$$

This is called the conservation of charge. [Hint: Start with $\nabla \cdot (\nabla \times \mathbf{B}) = 0$.]

Solution:

$$\begin{aligned}
0 &= \nabla \cdot (\nabla \times \mathbf{B}) \\
&= \nabla \cdot \left(\mathbf{j} + \frac{\partial \mathbf{E}}{\partial t} \right) \\
&= \nabla \cdot \mathbf{j} + \frac{\partial}{\partial t} (\nabla \cdot \mathbf{E}) \\
&= \nabla \cdot \mathbf{j} + \frac{\partial}{\partial t} \rho.
\end{aligned}$$

Exercise 1.4. [Bonus] Putting the physical constants back in Maxwell's equations with $\rho = 0, \mathbf{j} = 0$ leads to the following two equations (in vacuum):

$$\nabla^2 \mathbf{E} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \tag{1.13}$$
$$\nabla^2 \mathbf{B} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0$$

In vacuum then, each component of $\mathbf{E}$ and $\mathbf{B}$ satisfies a 3-dimensional wave equation $\nabla^2 f = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$ with $v = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$. Do you recognize the numerical value of this wave speed? (Note that we did not specify how our coordinate system was moving with respect to the electromagnetic wave).

If we consider a different coordinate system, moving relative to the first one via

$$t' = t \tag{1.14}$$

$$x' = x - ut$$

(think of this as the coordinate system of a person sitting on a train moving at speed u relative to the wave). What happens to the wave equations in the new coordinate system? (Hint: It's enough to look at just one component).

In contrast, what happens for the following transformation of the coordinates (this is called a boost)

$$\begin{aligned} t' &= \gamma(t - \frac{u}{c^2}x) \\ x' &= \gamma(x - ut) \end{aligned} \tag{1.15}$$

where $\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$?

Historical note: This is what lead Einstein to invent the coordinate transformations above (an example of a Lorenz transformation) and formulate his special theory of relativity. This means that Maxwell's equations have a lot to tell us about how different observers in the universe experience space and time.

Other strange things are worth noting: **E** and **B** also transform under a boost meaning that electric and magnetic fields are themselves not fundamental physical objects. We'll return to this.

Solution: The constant v is the speed of light in a vacuum given by $v = c \simeq 2.998 \times 10^8 m/s$.

Using the coordinate transformation in (1.14) we find that

$$\begin{aligned} \frac{\partial}{\partial t} &= \frac{\partial x'}{\partial t}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} \\ &= -u\frac{\partial}{\partial x'} + \frac{\partial}{\partial t'} \\ \frac{\partial}{\partial x} &= \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial x}\frac{\partial}{\partial t'} \\ &= \frac{\partial}{\partial x'} \\ \frac{\partial}{\partial y} &= \frac{\partial}{\partial y'} \\ \frac{\partial}{\partial z} &= \frac{\partial}{\partial z'} \end{aligned}$$

We compute that

$$\partial_x^2 + \partial_y^2 + \partial_z^2 - \frac{1}{c^2}\partial_t^2 = \partial_{x'}^2 + \partial_{y'}^2 + \partial_{z'}^2 - \frac{1}{c^2}\partial_{t'}^2 + \left(-\frac{u^2}{c^2}\partial_{x'}^2 + \frac{2u}{c^2}\partial_{x'}\partial_{t'}\right)$$

This is because

$$\left(\partial_x^2 - \frac{1}{c^2}\partial_t^2\right) f = \partial_{x'}^2 f - \frac{1}{c^2}\left(-u\frac{\partial}{\partial x'} + \frac{\partial}{\partial t'}\right)\left(-u\frac{\partial f}{\partial x'} + \frac{\partial f}{\partial t'}\right)$$
$$= \left(1 - \frac{u^2}{c^2}\right)\partial_{x'}^2 f + \frac{2u}{c^2}\partial_{x'}\partial_{t'} f - \frac{1}{c^2}\partial_{t'}^2 f.$$

Hence, Maxwell's equations in the coordinates (t, x, y, z) are not the same as in coordinates (t', x', y', z').

Using the coordinate transformation in (1.15) we find that

$$\begin{aligned}
\frac{\partial}{\partial t} &= \frac{\partial x'}{\partial t}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} \\
&= -\gamma u\frac{\partial}{\partial x'} + \gamma\frac{\partial}{\partial t'} \\
\frac{\partial}{\partial x} &= \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial x}\frac{\partial}{\partial t'} \\
&= \gamma\frac{\partial}{\partial x'} - \frac{\gamma u}{c^2}\frac{\partial}{\partial t'} \\
\frac{\partial}{\partial y} &= \frac{\partial}{\partial y'} \\
\frac{\partial}{\partial z} &= \frac{\partial}{\partial z'}
\end{aligned}$$

We compute that

$$\partial_x^2 + \partial_y^2 + \partial_z^2 - \frac{1}{c^2}\partial_t^2 = \partial_{x'}^2 + \partial_{y'}^2 + \partial_{z'}^2 - \frac{1}{c^2}\partial_{t'}^2.$$

This is because

$$\left(\partial_x^2 - \frac{1}{c^2}\partial_t^2\right) f = \left(\gamma\frac{\partial}{\partial x'} - \frac{\gamma u}{c^2}\frac{\partial}{\partial t'}\right)\left(\gamma\frac{\partial f}{\partial x'} - \frac{\gamma u}{c^2}\frac{\partial f}{\partial t'}\right) - \frac{1}{c^2}\left(-\gamma u\frac{\partial}{\partial x'} + \gamma\frac{\partial}{\partial t'}\right)\left(-\gamma u\frac{\partial f}{\partial x'} + \gamma\frac{\partial f}{\partial t'}\right)$$
$$= \left(\gamma^2 - \gamma^2\frac{u^2}{c^2}\right)\partial_{x'}^2 f + \left(-2\frac{\gamma^2 u}{c^2} - \frac{1}{c^2}(-\gamma^2 u)\right)\partial_{x'}\partial_{t'} + \left(\frac{\gamma^2 u^2}{c^4} - \frac{1}{c^2}\gamma^2\right)\partial_{t'}^2$$
$$= \partial_{x'}^2 f - \frac{1}{c^2}\partial_{t'} f,$$

since $\gamma^2(1 - \frac{u^2}{c^2}) = 1$. Hence, Maxwell's equations in the coordinates (t, x, y, z) and (t', x', y', z') are the same.

2. Lecture 2: Manifolds & Vector Fields (LF)

[(Tuesday, July 2, 2019)]

2.1. **Manifolds.** Given a topological space X and an open set $U \subset X$, we define a **chart** to be a continuous function $\varphi : U \to \mathbb{R}^n$ with a continuous inverse. As long as we work "in the chart φ" we can pretend we are working in $\mathbb{R}^n$. For example, if we have a function $f : U \to \mathbb{R}$, we can turn it into a function on $\mathbb{R}^n$ by using $f \circ \varphi^{-1} : \mathbb{R}^n \to \mathbb{R}$. (This function is only defined on $\varphi(U) \subset \mathbb{R}^n$.)

Definition 2.1. An n-dimensional **manifold** is a topological space M equipped with charts $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ where U_α are open sets covering M such that the **transition functions** $\varphi_\alpha \circ \varphi_\beta^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ are each a smooth function where it is defined. Such a collection of charts is called an **atlas.**

2.2. **Classification of 2-dimensional manifolds.**

2.3. **Vector Fields.**

Definition 2.2. A **vector field** on M is a function

$$\begin{aligned} v : C^\infty(M) &\to C^\infty(M) \\ f &\mapsto v[f] \end{aligned}$$

such that for all $f, g \in C^\infty(M)$,

- $v[f+g] = v[f] + v[g]$
- $v[\alpha f] = \alpha v[f]$ for a constant $\alpha \in \mathbb{R}$
- $v[fg] = v[f]g + fv[g]$ "Leibniz rule"

In a local patch of M with coordinates $(x_1, \cdot, x_n)$, any vector field can be written

$$v = v^1 \frac{\partial}{\partial x_1} + \cdots + v^n \frac{\partial}{\partial x_n} \tag{2.1}$$

for $v_1, \cdots, v_n \in C^\infty(M)$.

Exercises.

Exercise 2.3. [Straightforward] Consider the following vector fields and functions on $\mathbb{R}^2$.

$$v = x_1 \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial x_2}, \qquad f(x_1, x_2) = x_1 x_2^2, \qquad g(x_1, x_2) = -x_2.$$

Verify that the Leibniz rule holds for $v[fg] = fv[g] + gv[f]$.

Solution: We compute:

$$\begin{aligned} v[f] &= x_1 \frac{\partial}{\partial x_1}[f] + 2x_2 \frac{\partial}{\partial x_2}[f] = x_1 \left(x_2^2\right) + 2x_2 \left(2x_1 x_2\right) \\ v[g] &= x_1 \frac{\partial}{\partial x_1}[g] + 2x_2 \frac{\partial}{\partial x_2}[g] = x_1 (0) + 2x_2 (-1) \\ v[fg] &= x_1 \frac{\partial}{\partial x_1}[-x_1 x_2^3] + 2x_2 \frac{\partial}{\partial x_2}[-x_1 x_2^3] \end{aligned}$$

$$
\begin{aligned}
&= x_1\left(-x_2^3\right) + 2x_2\left(-3x_1x_2^2\right) \\
&= -7x_1x_2^3
\end{aligned}
$$

We now check the Leibniz rule:

$$
\begin{aligned}
fv[g] + gv[f] &= \left(x_1x_2^2\right)\left(x_1(0) + 2x_2(-1)\right) + (-x_2)\left(x_1\left(x_2^2\right) + 2x_2\left(2x_1x_2\right)\right) \\
&= -2x_1x_2^3 - x_2\left(5x_1x_2^2\right) \\
&= -7x_1x_2^3
\end{aligned}
$$

which is indeed equal to $v[fg]$.

Exercise 2.4. [Straightforward] Compute $\frac{\partial}{\partial\theta}$ on in Cartesian coordinates on $\mathbb{R}^2 - \{0\}$ using the fact that

$$
\frac{\partial}{\partial\theta} = \frac{\partial x}{\partial\theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\theta}\frac{\partial}{\partial y}. \tag{2.2}
$$

Solution:

$$
\begin{aligned}
\frac{\partial}{\partial\theta} &= \frac{\partial x}{\partial\theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\theta}\frac{\partial}{\partial y} \\
&= \frac{\partial(r\cos\theta)}{\partial\theta}\frac{\partial}{\partial x} + \frac{\partial(r\sin\theta)}{\partial\theta}\frac{\partial}{\partial y} \\
&= -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y} \\
&= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}
\end{aligned}
$$

Exercise 2.5. Let $\mathbb{S}^2$ be the unit sphere $x^2 + y^2 + z^2 = 1$ sitting inside $\mathbb{R}^3$. In this exercise, we'll show that the $\mathbb{S}^2$ is a 2-dimensional manifold by describing two open sets:

- $U_N = \mathbb{S}^2 - \{N\}$, where $N = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is the north pole, and
- $U_S = \mathbb{S}^2 - \{S\}$, where $S = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$ is the south pole,

and the charts $\psi_N : U_N \to \mathbb{R}^2$ and $\psi_S : U_S \to \mathbb{R}^2$ coming from stereographic projection.

Stereographic projection from the north pole maps the sphere minus the north pole to the equatorial plane as follows: for any point $p = (x, y, z)^T$ on $\mathbb{S}^2$, there is a unique line through the north pole N and the point p. This line intersects the plane $z = 0$ in exactly one point p'. Then stereographic projection σ is the map

$$\begin{aligned} \psi_N : \mathbb{S}^2 - \{N\} &\to \mathbb{R}^2 \\ p &\mapsto p'. \end{aligned} \tag{2.3}$$

In coordinates,

$$\psi_N : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} X \\ Y \end{pmatrix} = \frac{1}{1-z} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad z \neq 1,$$

We can similarly do stereographic projection from the south pole S and get a map

$$\psi_S : \mathbb{S}^2 - \{S\} \to \mathbb{R}^2 \tag{2.4}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} X \\ Y \end{pmatrix} = \frac{1}{1+z} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{2.5}$$

(a) The inverse map of stereographic projection σ_N gives a parametrization of the unit sphere $\mathbb{S}^2 - \{N\}$. Write down this map

$$\psi_N^{-1} : \mathbb{R}^2 \to \mathbb{S}^2 - \{N\}. \tag{2.6}$$

(b) Compute the transition function $\psi_S \circ \psi_N^{-1} : \mathbb{R}^2 - \{0\} \to \mathbb{R}^2 - \{0\}$. Why is the natural domain $\mathbb{R}^2 - \{0\}$?

Solution: (a) Using (u, v) as the coordinate in the equatorial plane. We want to reverse the function $u = \frac{x}{1-z}$, $v = \frac{y}{1-z}$ in the previous question. We compute

$$u^2 + v^2 = \frac{x^2 + y^2}{(1-z)^2} = \frac{1-z^2}{(1-z)^2} = \frac{1+z}{1-z},$$

so $z = \frac{u^2+v^2-1}{u^2+v^2+1}$. Noticing $\frac{u}{v} = \frac{x}{y}$, and $x^2 + y^2 = 1 - z^2 = \frac{4u^2+4v^2}{(u^2+v^2+1)^2}$, we get $x = \frac{2u}{u^2+v^2+1}$, $y = \frac{2v}{u^2+v^2+1}$. Thus the map is

$$\psi_N^{-1}(u, v) = \begin{pmatrix} \frac{2u}{u^2+v^2+1} \\ \frac{2v}{u^2+v^2+1} \\ \frac{u^2+v^2-1}{u^2+v^2+1}. \end{pmatrix}$$

(b) Now we compute

$$\begin{aligned}
\psi_S \circ \psi_N^{-1} \begin{pmatrix} u \\ v \end{pmatrix} &= \psi_S \begin{pmatrix} \dfrac{2u}{u^2+v^2+1} \\ \dfrac{2v}{u^2+v^2+1} \\ \dfrac{u^2+v^2-1}{u^2+v^2+1}. \end{pmatrix} \\
&= \frac{1}{1+\left(\dfrac{u^2+v^2-1}{u^2+v^2+1}\right)} \begin{pmatrix} \dfrac{2u}{u^2+v^2+1} \\ \dfrac{2v}{u^2+v^2+1} \end{pmatrix} \\
&= \frac{1}{u^2+v^2+1+(u^2+v^2-1)} \begin{pmatrix} 2u \\ 2v \end{pmatrix} \\
&= \frac{1}{u^2+v^2+1+(u^2+v^2-1)} \begin{pmatrix} 2u \\ 2v \end{pmatrix} \\
&= \frac{1}{u^2+v^2} \begin{pmatrix} u \\ v \end{pmatrix}.
\end{aligned}$$

Exercise 2.6. [Bonus] Circles on the unit sphere $x^2 + y^2 + z^2 = 1$ are mapped under the stereographic projection

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} X \\ Y \end{pmatrix} = \frac{1}{1-z} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad z \neq 1,$$

to circles or lines in the equatorial plane. Prove this theorem using the following steps.

Recall that a circle on the surface of the sphere is the intersection of the sphere with a plane.

(a) Assume that $(x, y, z)^T$ lies on the plane $ax + by + cz = c$. Prove that $aX + bY = c$.
(b) Prove that $X^2 + Y^2 = \frac{1+z}{1-z}$.
(c) Suppose now that $(x, y, z)^T$ lies on the plane $ax + by + cz = 1 + c$. Use part (b) to show that in this case (X, Y) satisfies the equation of a circle

$$(X-a)^2 + (Y-b)^2 = a^2 + b^2 - 2c - 1. \tag{†}$$

It remains to show two things: that we have taken into consideration all circles on the sphere, and that the circle defined in part (c) really is a circle.

(d) Consider the plane $ax + by + cz = d$. If $c \neq d$, show that the equation of the plane may be rewritten so that $d - c = 1$. Argue that all planes in $\mathbb{R}^3$ are considered in parts (a) and (c).

(e) Assume that the plane $ax+by+cz=d$ intersects the unit sphere. By considering the unit normal vector $\frac{1}{\sqrt{a^2+b^2+c^2}}(a,b,c)^T$ to the plane, explain why we must have
$$\|d\| \leq \sqrt{a^2+b^2+c^2}.$$
What does equality in this formula mean? Use this to conclude that the right hand side of (†) is non-negative.

(f) Use parts (a) through (e) to argue that any circle on the surface of the sphere is projected to a circle or a line under the stereographic projection and that conversely every circle and line in the equatorial plane arises this way.

Solution:

(a)
$$aX+bY=\frac{ax}{1-z}+\frac{by}{1-z}=\frac{ax+by}{1-z}=\frac{c-cz}{1-z}=c.$$
(b)
$$X^2+Y^2=\frac{x^2+y^2}{(1-z)^2}=\frac{1-z^2}{(1-z)^2}=\frac{1+z}{1-z}.$$
(c)
$$\begin{aligned}(X-a)^2+(Y-b)^2 &= X^2+Y^2-2(aX+bY)+a^2+b^2\\ &= \frac{1+z}{1-z}-2\frac{1+c-cz}{1-z}+a^2+b^2\\ &= a^2+b^2-2c-1.\end{aligned}$$

(d) If $c \neq d$, divide the equation by $d-c$ to get $a'x+b'y+c'z=d'$. It represents the same plane, and $d'-c'=1$. As every plane in $\mathbb{R}^3$ can be written as in the form $ax+by+cz=d$ for some a,b,c,d. Part (a) takes care of the case when $c-d$. The previous argument reduces any other case to the situation in part (c).

(e) Suppose (x_0,y_0,z_0) is a intersection point. Then we have $x_0^2+y_0^2+z_0^2=1$, and $ax_0+by_0+cz_0=d=\langle \begin{pmatrix} a\\ b\\ c\end{pmatrix}, \begin{pmatrix} x_0\\ y_0\\ z_0\end{pmatrix}\rangle$, where $\langle\cdot,\cdot\rangle$ denote the usual inner product on $\mathbb{R}^3$. Then, by Cauchy-Schwarz inequality, we have $|d| \leq \sqrt{a^2+b^2+c^2}\sqrt{x_0^2+y_0^2+z_0^2}=\sqrt{a^2+b^2+c^2}$. Notice, in part (c), $d=1+c$, so we have $(1+c)^2 \leq a^2+b^2+c^2$, so $0 \leq a^2+b^2-2c-1$.

(f) Any circle on the surface of the sphere can be viewed as the intersection of the sphere with a plane in $\mathbb{R}^3$. If this plane if of the form in part (a), (if you think about it, the condition in part (a) is saying that the plane passes the north pole $N=\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$) we get the

image is a line in the equatorial plane. If this plane is of the form in part (c), we get a circle in the equatorial plane.
Conversely, suppose we are given a line L in the equatorial plane, consider the plane passing L and the north pole N in $\mathbb{R}^3$. It intersects the sphere at some circle. Suppose we are given some circle in the equatorial plane, it can be described by a equation of the form $(X-a)^2+(Y-b)^2=r^2$, for some a,b,r. Then, by the result in part (c), let $c=\dfrac{r^2+1-a^2-b^2}{2}$, we get the circle is the image of some circle which is the intersection of the plane $ax+by+cz=1+c$ and the sphere.

3. Lecture 3: Differential 1-Forms & Exterior Derivative & Differential Forms & Changes of Coordinates(LF)

[(given Wednesday, July 3, 2019)]

3.1. **Differential 1-Forms, $\Omega^1(M)$.** In any local patch U that looks like $\mathbb{R}^n$, we define the 1-forms $\mathrm{d}x_1,\cdots,\mathrm{d}x_n$ by the properties

$$\mathrm{d}x_i\left(\frac{\partial}{\partial x_j}\right)=\begin{cases}1 & i=j\\ 0 & i\neq j.\end{cases} \tag{3.1}$$

It is a fact that any 1-form on any patch U that looks like $\mathbb{R}^n$ can be written

$$\alpha=f_1\mathrm{d}x_1+f_2\mathrm{d}x_2+\cdots+f_n\mathrm{d}x_n, \tag{3.2}$$

for some real-valued functions $f_1,\cdots,f_n:U\to\mathbb{R}$. Recall that any vector field on any patch U that looks like $\mathbb{R}^n$ can be written

$$v=g_1\frac{\partial}{\partial x_1}+g_2\frac{\partial}{\partial x_2}+\cdots+g_n\frac{\partial}{\partial x_n}, \tag{3.3}$$

for some real-valued functions $g_1,\cdots,g_n:U\to\mathbb{R}$. Then

$$\begin{aligned}\alpha(v) &= f_1\mathrm{d}x_1(v)+f_2\mathrm{d}x_2(v)+\cdots+f_n\mathrm{d}x_n(v)\\ &= f_1g_1+f_2g_2+\cdots+f_ng_n.\end{aligned} \tag{3.4}$$

Indeed, note that this final expression for $\alpha(v)$ is a function.

Note the similarity between this final expression and the dot product in $\mathbb{R}^n$ of the vector fields v and $w=f_1\frac{\partial}{\partial x_1}+f_2\frac{\partial}{\partial x_2}+\cdots+f_n\frac{\partial}{\partial x_n}$.

3.2. **Tangent space & cotangent space.**

3.3. **Exterior derivative,** d. Given a function f, there is a naturally associated 1-form $\mathrm{d}f$ such that $v[f]=\mathrm{d}f(v)$.

Let M be an n-dimensional manifold, and let $f \in C^\infty(M)$ be a smooth real-valued function on M. In any patch U that looks like $\mathbb{R}^n$, the 1-form $\mathrm{d}f$ is

$$\mathrm{d}f = \partial_{x_1} f \mathrm{d}x_1 + \partial_{x_2} f \mathrm{d}_{x_2} + \cdots + \partial_{x_n} f \mathrm{d}x_n. \tag{3.5}$$

3.4. **Differential k-forms, $\Omega^k(M)$.** On any patch U that looks like $\mathbb{R}^n$, the differential k-forms can be written in the basis

$$\{\mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2} \wedge \cdots \wedge \mathrm{d}x_{i_k}\}_{(i_1,\cdots,i_k)\in\mathcal{I}} \tag{3.6}$$

where the indicial set $\mathcal{I}$ is

$$\mathcal{I} = \{(i_1, \cdots, i_k) : 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}. \tag{3.7}$$

The wedge project "$\wedge$" is map

$$\begin{array}{rcl} \Omega^k(M) \times \Omega^\ell(M) & \to & \Omega^{k+\ell}(M) \\ (\alpha, \beta) & \mapsto & \alpha \wedge \beta. \end{array}$$

satisfying $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$.

View a differential k-form as a map from k copies of $\mathrm{Vect}(M)$ to $C^\infty(M)$. To describe the map, only need to give it in terms of this basis where each $\alpha_i \in \Omega^1(M)$:

$$\alpha_1 \wedge \cdots \wedge \alpha_k(v_1, \cdots, v_k) = \det \begin{pmatrix} \alpha_1(v_1) & \alpha_1(v_2) & \cdots & \alpha_1(v_k) \\ \alpha_2(v_1) & \alpha_2(v_2) & \cdots & \alpha_2(v_k) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_k(v_1) & \alpha_k(v_2) & \cdots & \alpha_k(v_k) \end{pmatrix} \tag{3.8}$$

Can extend the exterior derivative to a map

$$\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M).$$

In coordinates, if

$$\alpha = \sum_{I\in\mathcal{I}} = f_I \mathrm{d}x_I \qquad \text{where } \mathrm{d}x_I = \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k},$$

then

$$\mathrm{d}\alpha = \sum_{I\in\mathcal{I}} \mathrm{d}f_I \wedge \mathrm{d}x_I.$$

Proposition 3.1.

- $\mathrm{d}(\alpha + \beta) = \mathrm{d}\alpha + \mathrm{d}\beta$
- $\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^{\deg(\alpha)} \alpha \wedge \mathrm{d}\beta$
- $\mathrm{d}(\mathrm{d}\alpha) = 0$

Exercises.

Exercise 3.2. [Straightforward] Consider the vector fields

$$u = x\frac{\partial}{\partial x} - \frac{\partial}{\partial y}, \qquad v = y\frac{\partial}{\partial x} - xy\frac{\partial}{\partial z},$$

and the 2-forms

$$\begin{aligned} \alpha = y\mathrm{d}x \wedge \mathrm{d}y - z\mathrm{d}x \wedge \mathrm{d}z, \qquad \beta = 3\mathrm{d}x \wedge \mathrm{d}z - yz\mathrm{d}y \wedge \mathrm{d}z, \\ \gamma = z\mathrm{d}x \wedge \mathrm{d}y - y\mathrm{d}x \wedge \mathrm{d}z + z\mathrm{d}y \wedge \mathrm{d}z. \end{aligned} \tag{3.9}$$

Calculate $\alpha(u,v)$, $\beta(u,v)$, $\gamma(u,v)$.

Solution: We show one less step each time.

$$\begin{aligned} \alpha(u,v) &= y(\mathrm{d}x(u)\mathrm{d}y(v) - \mathrm{d}x(v)\mathrm{d}y(u)) - z(\mathrm{d}x(u)\mathrm{d}z(v) - \mathrm{d}x(v)\mathrm{d}z(u)) \\ &= y(0 - y(-1)) - z(x(-xy) - 0) \\ &= y^2 + x^2yz \\ \beta(u,v) &= 3(x(-xy) - y \cdot 0) \\ &= yz(-1(-xy) - 0) \\ &= -3x^2y - xy^2z \\ \gamma(u,v) &= yz + x^2y^2 + xyz \end{aligned}$$

Exercise 3.3. [Straightforward] Compute the following:

(1) $\mathrm{d}(x^2)$
(2) $\mathrm{d}\left(x\mathrm{d}y \wedge \mathrm{d}z + y\mathrm{d}z \wedge \mathrm{d}x + z\mathrm{d}x \wedge \mathrm{d}y\right)$
(3) $\mathrm{d}(x^2\mathrm{d}y + 3xy\mathrm{d}x)$

Solution:

$$\begin{aligned} \mathrm{d}(x^2) &= 2x\mathrm{d}x \\ \mathrm{d}(x\mathrm{d}y \wedge \mathrm{d}z + y\mathrm{d}z \wedge \mathrm{d}x + z\mathrm{d}x \wedge \mathrm{d}y) &= \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z + \mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}x + \mathrm{d}z \wedge \mathrm{d}z \wedge \mathrm{d}y \\ &= 2\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \\ \mathrm{d}(x^2\mathrm{d}y + 3xy\mathrm{d}x) &= 2x\mathrm{d}x \wedge \mathrm{d}y + 3\mathrm{d}(xy) \wedge \mathrm{d}x \\ &= 2x\mathrm{d}x \wedge \mathrm{d}y + 3x\mathrm{d}y \wedge \mathrm{d}x \\ &= 2x\mathrm{d}x \wedge \mathrm{d}y - 3x\mathrm{d}x \wedge \mathrm{d}y \\ &= -x\mathrm{d}x \wedge \mathrm{d}y. \end{aligned}$$

Exercise 3.4. [Straightforward] Let $f(x,y) = x^2 - y^2$. Compute the scalar function $\mathrm{d}f(v)$ where $v = 2x\frac{\partial}{\partial x} - 3y\frac{\partial}{\partial y}$.

Solution: We compute

$$\mathrm{d}f = 2x\mathrm{d}x - 2y\mathrm{d}y.$$

Hence,

$$\begin{array}{rcl} \mathrm{d}f(v) & = & 2x\mathrm{d}x(v) - 2y\mathrm{d}y(v) \\ 2x(2x) - 2y(-3y) & & \\ & = & 4x^2 + 6y^2. \end{array}$$

Alternatively, $\mathrm{d}f(v) = v[f] = 4x^2 + 6y^2$.

Exercise 3.5. Given $\alpha \in \Omega^1(M)$, define a differential operator d_α by

$$\begin{array}{rcl} \mathrm{d}_\alpha : \Omega^p(M) & \to & \Omega^{p+1}(M) \\ \omega & \mapsto & \mathrm{d}\omega + \alpha \wedge \omega. \end{array}$$

Now, take $M = \mathbb{R}$ with parameter t and $\alpha = c\mathrm{d}t$. Find all functions $\omega \in \Omega^0(\mathbb{R})$ such that

$$\mathrm{d}_\alpha \omega = 0. \tag{3.10}$$

Solution:

$$0 = \mathrm{d}_\alpha \omega = \partial_t \omega \mathrm{d}t + c\omega \mathrm{d}t = (\partial_t \omega + c\omega)\mathrm{d}t. \tag{3.11}$$

Thus $\omega = Ce^{-ct}$ for $C \in \mathbb{R}$.

Exercise 3.6. We can extend d to an operator mapping $\Omega^k(M)$-valued n-vectors to $\Omega^{k+1}(M)$-valued n-vectors.

$$\mathrm{d}\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} = \begin{pmatrix} \mathrm{d}\beta_1 \\ \vdots \\ \mathrm{d}\beta_n \end{pmatrix}$$

Similarly, the $\Omega^\ell(M)$-valued $n \times n$ matrix A maps $\Omega^k(M)$-valued n-vectors to $\Omega^{k+\ell}(M)$-valued n-vectors by

$$A \wedge \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^n A_{1j} \wedge \beta_j \\ \vdots \\ \sum_{j=1}^n A_{nj} \wedge \beta_j \end{pmatrix}$$

Let s be an $\Omega^0(M)$-valued n-vector, and let A be an $\Omega^1(M)$-valued $n \times n$ matrix. Define an operator d_A by

$$\mathrm{d}_A s = \mathrm{d}s + A \wedge s. \tag{3.12}$$

Show that

$$\mathrm{d}_A(\mathrm{d}_A s) = F_A s, \tag{3.13}$$

where F_A is an $\Omega^2(M)$-valued $n \times n$ matrix equal

$$F_A = \mathrm{d}A + A \wedge A. \tag{3.14}$$

In our expression for F_A, note that we've extended d and $A\wedge$ to act on $\Omega^1(M)$-valued matrices. [Hint: Write it out completely when $n = 2$. The notation is more cumbersome for general n, though it is conceptually the same.] [We'll use this later! The operator d_A is a "connection on a vector bundle" and F_A is its curvature.]

Solution: We present the solution for $n = 2$. Let A be equal to

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22}. \end{pmatrix}$$

Then

$$\begin{aligned}
\mathrm{d}_A(\mathrm{d}_A s) &= \mathrm{d}_A(\mathrm{d}s + A \wedge s) \\
&= \mathrm{d}^2 s + A \wedge \mathrm{d}s + \mathrm{d}(A \wedge s) + A \wedge A \wedge s \\
&= A \wedge \mathrm{d}s + \mathrm{d}(A \wedge s) + A \wedge A \wedge s \\
&= \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22}. \end{pmatrix} \wedge \begin{pmatrix} \mathrm{d}s_1 \\ \mathrm{d}s_2 \end{pmatrix} + \mathrm{d}\begin{pmatrix} \alpha_{11}s_1 + \alpha_{12}s_2 \\ \alpha_{21}s_1 + \alpha_{22}s_2 \end{pmatrix} + A \wedge A \wedge s \\
&= \begin{pmatrix} \alpha_{11} \wedge \mathrm{d}s_1 + \alpha_{12} \wedge \mathrm{d}s_2 + \mathrm{d}\alpha_{11}s_1 - \alpha_{11} \wedge \mathrm{d}s_1 + \mathrm{d}\alpha_{12}s_2 - \alpha_{12}\mathrm{d}s_1 \\ \alpha_{21} \wedge \mathrm{d}s_1 + \alpha_{22} \wedge \mathrm{d}s_2 + \mathrm{d}\alpha_{21}s_1 - \alpha_{21} \wedge \mathrm{d}s_1 + \mathrm{d}\alpha_{22}s_2 - \alpha_{22}\mathrm{d}s_1 \end{pmatrix} + A \wedge A \wedge s \\
&= \begin{pmatrix} \mathrm{d}\alpha_{11}s_1 + \mathrm{d}\alpha_{12}s_2 \\ \mathrm{d}\alpha_{21}s_1 + \mathrm{d}\alpha_{22}s_2 \end{pmatrix} + A \wedge A \wedge s \\
&= \begin{pmatrix} \mathrm{d}\alpha_{11} & \mathrm{d}\alpha_{12} \\ \mathrm{d}\alpha_{21} & \mathrm{d}\alpha_{22} \end{pmatrix} \wedge \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} + A \wedge A \wedge s \\
&= (\mathrm{d}A + A \wedge A)s
\end{aligned}$$

Exercise 3.7. [Bonus] Let $f(x,y) = u(x,y) + iv(x,y)$ be a complex-valued function $f : \mathbb{R}^2 \to \mathbb{C}$ where u, v are real-valued. A function f is said to be *holomorphic* if $\frac{\partial}{\partial \bar{z}} f = \mathrm{d}f\left(\frac{\partial}{\partial \bar{z}}\right) = 0$, where $z, \bar{z}$ are viewed as co-ordinates on $\mathbb{C}$. Writing $z = x + iy$, prove that f is holomorphic if and only if u, v satisfy the *Cauchy-Riemann equations*:

$$u_x = v_y, \qquad v_x = -u_y.$$

Solution: Note that since $z = x + \mathrm{i}y$, $\bar{z} = x - \mathrm{i}y$, then

$$\begin{aligned} x &= \frac{z+\bar{z}}{2} \\ y &= \frac{z-\bar{z}}{2\mathrm{i}}. \end{aligned}$$

Then we compute:

$$\begin{aligned} 0 = \mathrm{d}f\left(\frac{\partial}{\partial \bar{z}}\right) &= (\mathrm{d}u + \mathrm{i}\mathrm{d}v)\left(\frac{\partial x}{\partial \bar{z}}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \bar{z}}\frac{\partial}{\partial y}\right) \\ &= (u_x \mathrm{d}x + u_y \mathrm{d}y + \mathrm{i}v_x \mathrm{d}x + \mathrm{i}v_y \mathrm{d}y)\left(\frac{1}{2}\frac{\partial}{\partial x} - \frac{1}{2\mathrm{i}}\frac{\partial}{\partial y}\right) \\ &= u_x\frac{1}{2} - u_y\frac{1}{2\mathrm{i}} + \mathrm{i}v_x\frac{1}{2} - \mathrm{i}v_y\frac{1}{2\mathrm{i}} \\ &= \frac{1}{2}\left((u_x - v_y) + \mathrm{i}(v_x + u_y)\right). \end{aligned}$$

If a complex-valued function is zero, then it's real and imaginary parts are separately zero, i.e.

$$u_x - v_y = 0 \qquad v_x + u_y = 0. \tag{3.15}$$

These are the Cauchy-Riemann equations.

4. Lecture 4: Metrics & Hodge Star (LF)

[(Friday, July 5, 2019)]

4.1. Metrics.

Definition 4.1. A **semi-Riemannian metric** on a vector space V is a map

$$g : V \times V \to \mathbb{R} \tag{4.1}$$

that is

- bilinear, or linear in each slot:

$$\begin{aligned} g(cv + v', w) &= cg(v, w) + g(v', w) \\ g(v, cw + w') &= cg(v, w) + g(v, w') \end{aligned}$$

- symmetric: $g(v, w) = g(w, v)$
- nondegenerate: If $g(v, w) = 0$ for all $w \in V$, then $v = 0$.

Semi-Riemannian metrics on the vector space $\mathbb{R}^n$ are given by symmetric matrices with $\det M \neq 0$. The metric is given by

$$g(v, w) = w^T M v. \tag{4.2}$$

We define the **signature** of a metric g on $\mathbb{R}^n$ to (σ_+, σ_-) where σ_+ is the number of positive eigenvalues of the associated metric M and σ_- is the number of negative eigenvalues of M.

Definition 4.2. A semi-Riemannian metric g on an n-dimensional manifold is a family of semi-Riemannian metrics

$$g_p : T_pM \times T_pM \to \mathbb{R} \qquad p \in M$$

such that for every pair of vector fields v, w on M the function $f : M \to \mathbb{R}$ defined by

$$f(p) = g_p(v|_p, w|_p)$$

is smooth.

Example 4.3. *A general metric on patch U of a 2-dimensional manifold can be written*

$$g = E\mathrm{d}x^2 + 2F\mathrm{d}x\mathrm{d}y + G\mathrm{d}y^2,$$

where $\mathrm{d}x^2, \mathrm{d}x\mathrm{d}y, \mathrm{d}y^2$ each map a pair of vector fields to a function defined by

$$\begin{aligned}
\mathrm{d}x^2(v,w) &= \mathrm{d}x(v)\mathrm{d}x(w)\\
\mathrm{d}x\mathrm{d}y(v,w) &= \frac{1}{2}\left(\mathrm{d}x(v)\mathrm{d}y(w) + \mathrm{d}y(v)\mathrm{d}x(w)\right)\\
\mathrm{d}y^2(v,w) &= \mathrm{d}y(v)\mathrm{d}y(w).
\end{aligned}$$

(More formally, we'd call g a "symmetric 2-tensor.") The associated matrix is

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}. \tag{4.3}$$

Example 4.4. *The Euclidean metric on $\mathbb{R}^n$ is written*

$$g = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \cdots + \mathrm{d}x_n^2. \tag{4.4}$$

Example 4.5. *The Lorentzian metric on $\mathbb{R}^{n,1}$ is*

$$g = -\mathrm{d}t^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \cdots + \mathrm{d}x_n^2. \tag{4.5}$$

4.2. **Lengths.**

4.3. **Volumes.** Let M be an n-dimensional manifold with Riemannian metric g. In a local coordinate patch, let $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$. Then the **volume form** is the following nowhere-vanishing n-form:

$$\mathrm{dvol} = \sqrt{|\det g_{ij}|}\mathrm{d}x^1 \wedge \cdots \mathrm{d}x^n. \tag{4.6}$$

4.4. **Hodge star.** There are three basis operations on differential k-forms: the wedge product "$\wedge$", the exterior derivative "d", and the Hodge star "$\star$".

The Hodge star operator on an n-dimensional manifold M with Riemannian metric g is a map

$$\star : \Omega^p(M) \to \Omega^{n-p}(M). \tag{4.7}$$

If we are using coordinates such that $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n}$ are orthonormal, then the Hodge star operator is defined by

$$\mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2} \wedge \cdots \mathrm{d}x_{i_k} = (-1)^\sigma \mathrm{d}x_{i_{k+1}} \wedge \cdots \mathrm{d}x_{i_n},$$

where $(i_1, i_2, \cdots, i_n)$ is a permutation of $(1, 2, \cdots, n)$ and σ is the sign of this permutation.

It is easy to verify that $\star^2\alpha = (-1)^{n(k-n)}\alpha$ for $\alpha \in \Omega^k(M)$.

Example 4.6. *In $\mathbb{R}^3$ with the Euclidean metric $g = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$, the Hodge star operator acts as follows: On $\Omega^0(\mathbb{R}^3)$,*

$$\star 1 = \pm \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \tag{4.8}$$

On $\Omega^1(\mathbb{R}^3)$,

$$\star \mathrm{d}x = \mathrm{d}y \wedge \mathrm{d}z \qquad \star \mathrm{d}y = \mathrm{d}z \wedge \mathrm{d}x \qquad \star \mathrm{d}z = \mathrm{d}x \wedge \mathrm{d}y$$

On $\Omega^2(\mathbb{R}^3)$,

$$\star \mathrm{d}x \wedge \mathrm{d}y = \qquad \star \mathrm{d}x \wedge \mathrm{d}z = \qquad \star \mathrm{d}y \wedge \mathrm{d}z =$$

On $\Omega^3(\mathbb{R}^3)$,

$$\star(\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z)$$

4.5. **Application to vector calculus in $\mathbb{R}^3$.** The standard vector calculus operations of div, grad and curl in $(\mathbb{R}^3, g_{\mathrm{Euc}})$ are closely related to 'd'. For example, the curl of a vector field $\mathbf{v} = (\alpha_x, \alpha_y, \alpha_z)$ is

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \alpha_x & \alpha_y & \alpha_z \end{vmatrix} = \left(\frac{\partial \alpha_z}{\partial y} - \frac{\partial \alpha_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial \alpha_x}{\partial z} - \frac{\partial \alpha_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial \alpha_y}{\partial x} - \frac{\partial \alpha_x}{\partial y}\right)\mathbf{k},$$

while the exterior derivative of the 1-form $\alpha = \alpha_x \mathrm{d}x + \alpha_y \mathrm{d}y + \alpha_z \mathrm{d}z$ is

$$\mathrm{d}\alpha = \left(\frac{\partial \alpha_y}{\partial x} - \frac{\partial \alpha_x}{\partial y}\right)\mathrm{d}x \wedge \mathrm{d}y - \left(\frac{\partial \alpha_x}{\partial z} - \frac{\partial \alpha_z}{\partial x}\right)\mathrm{d}x \wedge \mathrm{d}z + \left(\frac{\partial \alpha_z}{\partial y} - \frac{\partial \alpha_y}{\partial z}\right)\mathrm{d}y \wedge \mathrm{d}z.$$

Comparing coefficients gives part of the following table: start on any line and compare what d does to the form with what the corresponding vector calculus operation does to the object on the right hand side.

$$
\begin{array}{ccc}
\textbf{Forms} & & \textbf{Traditional vector fields} \\[1em]
\text{function} f & \longleftrightarrow & f \\
\big\downarrow \mathrm{d} & & \text{grad} \big\downarrow \nabla \bullet \\
\text{1-form } \alpha_x \mathrm{d}x + \alpha_y \mathrm{d}y + \alpha_z \mathrm{d}z & \longleftrightarrow & \alpha_x \mathbf{i} + \alpha_y \mathbf{j} + \alpha_z \mathbf{k} \\
\big\downarrow \mathrm{d} & & \text{curl} \big\downarrow \nabla \times \bullet \\
\text{2-form } \beta_1 \mathrm{d}y \wedge \mathrm{d}z + \beta_2 \mathrm{d}z \wedge \mathrm{d}x + \beta_3 \mathrm{d}x \wedge \mathrm{d}y & \longleftrightarrow & \beta_1 \mathbf{i} + \beta_2 \mathbf{j} + \beta_3 \mathbf{k} \\
\big\downarrow \mathrm{d} & & \text{div} \big\downarrow \nabla \cdot \bullet \\
\text{3-form } \gamma \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z & \longleftrightarrow & \text{function } \gamma
\end{array}
$$

The single 'd' operator is grad, div and curl all in one!

The differential form notation has two distinct advantages over traditional vector calculus: (1) it works in all co-ordinate systems and all dimensions; (2) the result $\mathrm{d}^2 = 0$ translates to the 2 theorems,

$$\nabla \times (\nabla f) = 0, \qquad \nabla \cdot (\nabla \times \mathbf{v}) = 0.$$

4.6. **Rewriting static Maxwell's equations.** From the above discussion, there are three ways to interpret an vector in $\mathbb{R}^3$:

- as a vector field
- as a differential 1-form
- as a differential 2-form.

Maxwell's equations are particularly clean if we interpret the electric vector field $\mathbf{E} = (E_x, E_y, E_z)$ as a 1-form on $\mathbb{R}^3$ as

$$E = E_x \mathrm{d}x + E_y \mathrm{d}y + E_z \mathrm{d}z \tag{4.9}$$

and if we interpret the magnetic vector field $\mathbf{B}$ as a 2-form on $\mathbb{R}^3$

$$B = B_x \mathrm{d}y \wedge \mathrm{d}z + B_y \mathrm{d}z \wedge \mathrm{d}x + B_z \mathrm{d}x \wedge \mathrm{d}y. \tag{4.10}$$

Then

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{E} = 0 \tag{4.11}$$

is equivalent to

$$\mathrm{d}B = 0 \qquad \mathrm{d}E = 0. \tag{4.12}$$

The second pair of static Maxwell's equations is

$$\nabla \cdot \mathbf{E} = \rho \qquad \nabla \times \mathbf{B} = \mathbf{j}. \tag{4.13}$$

This ends up being equivalent to

$$\star \mathrm{d} \star E = \rho \qquad \star \mathrm{d} \star B = j, \tag{4.14}$$

where $j = j_x \mathrm{d}x + j_y \mathrm{d}y + j_z \mathrm{d}z$ is determined from the components of $\mathbf{j} = (j_x, j_y, j_z)$.

Exercises.

Exercise 4.7. [Straightforward]Let $e_1, \cdots, e_n$ be the usual orthonormal basis of $\mathbb{R}^n$ and $x_1, \cdots, x_n$ the dual basis of $(\mathbb{R}^n)^*$. Compute $\star\alpha$ for the following differential forms:

(a) $\alpha = 9\mathrm{d}x_1 \wedge \mathrm{d}x_3 \in \Omega^2(\mathbb{R}^4)$
(b) $\alpha = \mathrm{d}x_1 + \mathrm{d}x_2 + x_1 x_2^2 \mathrm{d}x_3 \in \Omega^1(\mathbb{R}^3)$

Solution: (a) We know that

$$\star(\mathrm{d}x_1 \wedge \mathrm{d}x_3) = (-1)^\sigma \mathrm{d}x_2 \wedge \mathrm{d}x_4,$$

where σ is the sign of the permutation $(1, 3, 2, 4)$. Swapping 2 and 3, see see σ is odd, hence

$$\star(\mathrm{d}x_1 \wedge \mathrm{d}x_3) = -\mathrm{d}x_2 \wedge \mathrm{d}x_4.$$

(b) From lecture

$$\star\mathrm{d}x_1 = \mathrm{d}x_2 \wedge \mathrm{d}x_3 \qquad \star\mathrm{d}x_2 = \mathrm{d}x_3 \wedge \mathrm{d}x_1 \qquad \star\mathrm{d}x_3 = \mathrm{d}x_1 \wedge \mathrm{d}x_2. \tag{4.15}$$

Hence, since $\star$ is linear,

$$\star\alpha = \mathrm{d}x_2 \wedge \mathrm{d}x_3 - \mathrm{d}x_1 \wedge \mathrm{d}x_3 + x_1 x_2^2 \mathrm{d}x_1 \wedge \mathrm{d}x_2. \tag{4.16}$$

Exercise 4.8. Show that taking the wedge product of 1-forms translates to taking the cross product of vectors, i.e. if $\alpha = v_x \mathrm{d}x + v_y \mathrm{d}y + v_z \mathrm{d}z$ and $\beta = w_x \mathrm{d}x + w_y \mathrm{d}y + w_z \mathrm{d}z$, then

$$\star(\alpha \wedge \beta) = q_x \mathrm{d}x + q_y \mathrm{d}y + q_z \mathrm{d}z \tag{4.17}$$

for

$$\mathbf{q} = \mathbf{v} \times \mathbf{w}. \tag{4.18}$$

Solution: We compute

$$\mathbf{v} \times \mathbf{w} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \times \begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix}$$

$$= \begin{pmatrix} v_y w_z - v_z w_y \\ -(v_x w_z - v_z w_x) \\ v_x w_y - v_y w_x \end{pmatrix}.$$

On the other hand,

$$\begin{aligned} \star(\alpha \wedge \beta) &= \star\left((v_x w_y - v_y w_x)\mathrm{d}x \wedge \mathrm{d}y + (v_x w_z - v_z w_x)\mathrm{d}x \wedge \mathrm{d}z + (v_y w_z - v_z w_y)\mathrm{d}y \wedge \mathrm{d}z\right) \\ &= (v_x w_y - v_y w_x)\mathrm{d}z - (v_x w_z - v_z w_x)\mathrm{d}y + (v_y w_z - v_z w_y)\mathrm{d}x. \end{aligned}$$

Exercise 4.9. [Bonus] Consider $\mathbb{R}^4$ with the Euclidean metric.

(a) An differential 2-form β on $\mathbb{R}^4$ is called self-dual if $\star\beta = \beta$. What is the dimension of the space of self-dual differential 2-forms on $\mathbb{R}^4$. Find a basis of this space. `[Hint: Prove` $\alpha + \star\alpha$ `is self-dual.]`

(b) An differential 2-form γ on $\mathbb{R}^4$ is called anti-self-dual if $\star\gamma = -\gamma$. What is the dimension of the space of self-dual differential 2-forms on $\mathbb{R}^4$. Find a basis of this space.

(c) Suppose β is self-dual, and γ is anti-self-dual. Prove that $\langle\langle \beta, \gamma \rangle\rangle = 0$ where $\langle\langle \alpha_1, \alpha_2 \rangle\rangle = \star(\alpha_1 \wedge \star\alpha_2)$. Conclude that we have the orthogonal decomposition

$$\Omega^2(\mathbb{R}^4) = \{\text{self-dual 2-forms}\} \oplus \{\text{anti-self-dual 2-forms}\}.$$

Solution: (a) We first prove that $\alpha + \star\alpha$ is self-dual:

$$\begin{aligned} \star(\alpha + \star\alpha) &= \star\alpha + \star^2\alpha \\ &= \star\alpha + \alpha. \end{aligned}$$

Here we used that $\star^2 = 1$ on $\Omega^2(\mathbb{R}^4)$.

A basis for $\Omega^2(\mathbb{R}^2)$ is $\mathrm{d}x_1 \wedge \mathrm{d}x_2, \mathrm{d}x_1 \wedge \mathrm{d}x_3, \mathrm{d}x_1 \wedge \mathrm{d}x_4, \mathrm{d}x_2 \wedge \mathrm{d}x_3, \mathrm{d}x_2 \wedge \mathrm{d}x_4, \mathrm{d}x_3 \wedge \mathrm{d}x_4$. From this we can compute all the stars

$$\begin{aligned} \star\left(\mathrm{d}x_1 \wedge \mathrm{d}x_2\right) &= \mathrm{d}x_3 \wedge \mathrm{d}x_4 \\ \star\left(\mathrm{d}x_1 \wedge \mathrm{d}x_3\right) &= -\mathrm{d}x_2 \wedge \mathrm{d}x_4 \\ \star\left(\mathrm{d}x_1 \wedge \mathrm{d}x_4\right) &= \mathrm{d}x_2 \wedge \mathrm{d}x_3 \\ \star\left(\mathrm{d}x_2 \wedge \mathrm{d}x_3\right) &= -\mathrm{d}x_1 \wedge \mathrm{d}x_4 \\ \star\left(\mathrm{d}x_2 \wedge \mathrm{d}x_4\right) &= -\mathrm{d}x_1 \wedge \mathrm{d}x_3 \\ \star\left(\mathrm{d}x_3 \wedge \mathrm{d}x_4\right) &= -\mathrm{d}x_1 \wedge \mathrm{d}x_2 \end{aligned}$$

Hence, we form a basis for the space of self-dual forms in $\Omega^2(\mathbb{R}^4)$ by taking $\alpha + \star\alpha$ for every basis element of $\Omega^2(\mathbb{R}^2)$:

$$\mathrm{d}x_1 \wedge \mathrm{d}x_2 + \mathrm{d}x_3 \wedge \mathrm{d}x_4, \qquad \mathrm{d}x_1 \wedge \mathrm{d}x_3 - \mathrm{d}x_2 \wedge \mathrm{d}x_4 \qquad \mathrm{d}x_1 \wedge \mathrm{d}x_4 + \mathrm{d}x_2 \wedge \mathrm{d}x_3. \tag{4.19}$$

(b) Similarly, $\beta - \star\beta$ is anti-self-dual so a basis is

$$dx_1 \wedge dx_2 - dx_3 \wedge dx_4, \qquad dx_1 \wedge dx_3 + dx_2 \wedge dx_4 \qquad dx_1 \wedge dx_4 - dx_2 \wedge dx_3. \tag{4.20}$$

(c) We first observe that for any $\beta, \gamma \in \Omega^2(\mathbb{R}^4)$,

$$\langle\langle \beta, \gamma \rangle\rangle = \langle\langle \star\beta, \star\gamma \rangle\rangle$$

by checking this on a basis of $\Omega^2(\mathbb{R}^4)$. We can easily get part of the way as follows:

$$\begin{aligned}\langle\langle \star\beta, \star\gamma \rangle\rangle &= \star\left((\star\beta) \wedge \star(\star\gamma)\right)\\ &= \star\left(\star\beta \wedge \gamma\right)\\ &= \star\left(\gamma \wedge \star\beta\right)\\ &= \langle\langle \gamma, \beta \rangle\rangle\end{aligned}$$

Hence, now taking β to be self-dual and γ to be anti-self-dual,

$$\begin{aligned}\langle\langle \beta, \gamma \rangle\rangle &= \langle\langle \star\beta, \star\gamma \rangle\rangle\\ &= \langle\langle \beta, -\gamma \rangle\rangle\\ &= -\langle\langle \beta, \gamma \rangle\rangle\end{aligned}$$

Hence, $\langle\langle \beta, \gamma \rangle\rangle = 0$

Exercise 4.10. In coordinates, the Laplacian is

$$\Delta f = \star d \star df.$$

On $\mathbb{R}^2$ with Cartesian coordinates, $\Delta f = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f$. Compute Δf in polar coordinates on $\mathbb{R}^2$ using that $\star(dr \wedge d\theta) = r^{-1}$, $\star dr = r d\theta$ and $\star d\theta = -r^{-1} dr$.

Solution: We compute that

$$\begin{aligned}\star d \star df &= \star d \star (\partial_r f dr + \partial_\theta f d\theta)\\ &= \star d \left(\partial_r f r d\theta - r^{-1} \partial_\theta f dr\right)\\ &= \star \left(\partial_r^2 f r dr \wedge d\theta + \partial_r f dr \wedge d\theta - r^{-1} \partial_\theta^2 f \partial\theta \wedge dr\right)\\ &= \partial_r^2 f r (r^{-1}) + \partial_r f (r^{-1}) - r^{-1} \partial_\theta^2 f (-r^{-1})\\ &= \partial_r^2 f + r^{-1} \partial_r f + r^{-2} \partial_\theta^2 f\end{aligned}$$

5. Lecture 5: Maxwell's Equations Revisited (LF)

[(given Monday, July 8, 2019)]

5.1. **Electromagnetic potential.** It's best to view the electric field as a 1-form

$$E = E_x \mathrm{d}x + E_y \mathrm{d}y + E_z \mathrm{d}z \tag{5.1}$$

and the magnetic field as a 2-form

$$B = B_x \mathrm{d}y \wedge \mathrm{d}z + B_y \mathrm{d}z \wedge \mathrm{d}x + B_z \mathrm{d}x \wedge \mathrm{d}y. \tag{5.2}$$

Here $E_x, E_y, E_z, B_x, B_y, B_z$ are real-valued functions on $\mathbb{R}^3_{x,y,z} \times \mathbb{R}_t$.

We can combine both fields into a unified electromagnetic field F, which is a 2-form on $\mathbb{R}^4$ given by

$$F = B + E \wedge \mathrm{d}t. \tag{5.3}$$

$\mathbf{d}\boldsymbol{F} = \mathbf{0}$: Then the first pair of Maxwell's equations

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0 & (5.4)\\ \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0 & (5.5) \end{aligned}$$

is encoded in the single equation

$$\mathrm{d}F = 0.$$

To see this, on spacetimes like $S \times \mathbb{R}$, it makes sense to split d on $S \times \mathbb{R}$ as $\mathrm{d} = \mathrm{d}_S + \mathrm{d}_{\mathbb{R}} = \mathrm{d}_S + \mathrm{d}t \wedge \partial_t$, where d_S is the exterior derivative on S and $\mathrm{d}t$ is the exterior derivative on $\mathbb{R}_t$. Then

$$\mathrm{d}\alpha = \mathrm{d}_S\alpha + \mathrm{d}t \wedge \partial_t\alpha. \tag{5.6}$$

Then,

$$\begin{aligned} \mathrm{d}F &= \mathrm{d}(B + E \wedge \mathrm{d}t) & (5.7)\\ &= \mathrm{d}_S(B + E \wedge \mathrm{d}t) + \mathrm{d}t \wedge (\partial_t B + \partial_t E \wedge \mathrm{d}t)\\ &= \mathrm{d}_S B + \mathrm{d}_S E \wedge \mathrm{d}t + \partial_t B \wedge \mathrm{d}t\\ &= (\mathrm{d}_S B) + (\mathrm{d}_S E + \partial_t B) \wedge \mathrm{d}t \end{aligned}$$

Consequently, we see that $\mathrm{d}F = 0$ is equivalent to the equations

$$\mathrm{d}_S B = 0, \qquad \mathrm{d}_S E + \partial_t B = 0. \tag{5.8}$$

These are the equations

$$\nabla \cdot \mathbf{B} = \mathbf{0}, \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}. \tag{5.9}$$

$\star\mathbf{d}\star\boldsymbol{F} = \boldsymbol{J}$: The second pair of Maxwell's equations

$$\begin{aligned} \nabla \times \mathbf{B} &= \mathbf{j}\\ \nabla \cdot \mathbf{E} &= \rho \end{aligned}$$

is encoded in the equation

$$\star \,\mathrm{d} \star F = J, \tag{5.10}$$

where $J = j_x \mathrm{d}x + j_y \mathrm{d}y + j_z \mathrm{d}z - \rho \mathrm{d} \in \Omega^1(\mathbb{R}^{3,1})$.

Remark 5.1. It is a fact that $\star_{\mathbb{R}^3} E = \star_{\mathbb{R}^{3,1}}(E \wedge \mathrm{d}t)$ and $(-\star_{\mathbb{R}^3} B) \wedge \mathrm{d}t = \star_{\mathbb{R}^{3,1}} B$.

5.2. **Compatibility with special relativity.**

Definition 5.2. An **isometry** of vector spaces with respective metrics (V, g_V) and (W, g_W) is a bijective linear map $V \xrightarrow{f} W$ such that

$$g_V(v_1, v_2) = g_W(f(v_1), f(v_2)). \tag{5.11}$$

Definition 5.3. An **isometry** of manifolds (M, g) and (N, h) is a smooth map $M \xrightarrow{f} N$ with smooth inverse such that for all $p \in M$ and $v, w \in T_p M$,

$$g_p(v, w) = h_{f(p)}(f)_* v, f_* w). \tag{5.12}$$

The Lorentz group $SO(3,1)$ is the group of isometries of the vector space $\mathbb{R}^{3,1}$.

$$SO(3,1) = \{M \in \mathrm{Mat}_{4\times 4}(\mathbb{R}) : M^T G M = G \text{ and } \det M = 1\} \tag{5.13}$$

for

$$G := \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.14}$$

The Poincaré group is the group of isometries of the manifold $\mathbb{R}^{3,1}$ and includes both $SO(3,1)$ and translations.

We say that Maxwell's equations are "compatible with special relativity" since $\mathrm{d}F = 0$ and $\star \mathrm{d} \star F = J$ are preserved by the Poincaré group.

5.3. **Gauge symmetry.** As in Lecture 1, suppose the electric and magnetic fields **E**, **B** can be written in terms of a scalar "electric potential" φ and a vector "magnetic potential" **A** as

$$\mathbf{B} = \nabla \times \mathbf{A}, \qquad \mathbf{E} = -\nabla\varphi - \frac{\partial}{\partial t}\mathbf{A}. \tag{5.15}$$

Then the electromagnetic field is

$$F = \mathrm{d}\underbrace{\left(A_x \mathrm{d}x + A_y \mathrm{d}y + A_z \mathrm{d}z - \varphi \mathrm{d}t\right)}_{\widetilde{A}} \tag{5.16}$$

Recall, that $(\varphi, \mathbf{A})$ do not uniquely determine $(\mathbf{E}, \mathbf{B})$

Now we have

$$\widetilde{A}' = \widetilde{A} - \mathrm{d}\Gamma. \tag{5.17}$$

The fact that $\widetilde{A}'$ and $\widetilde{A}$ determine the same electromagnetic field F reduces to $\mathrm{d}^2 = 0$:

$$F' = \mathrm{d}\widetilde{A}' = \mathrm{d}\left(\widetilde{A} - \mathrm{d}\Gamma\right) = \mathrm{d}\widetilde{A} = F.$$

Exercises.

Exercise 5.4. [Straightforward] Let

$$B = -\sin(t-x)\mathrm{d}x \wedge \mathrm{d}z + \cos(t-x)\mathrm{d}x \wedge \mathrm{d}y.$$

Compute $\mathrm{d}B$. Compute $\star B$.

Solution:

$$\begin{aligned}
\mathrm{d}B &= -\mathrm{d}\left(\sin(t-x)\right) \wedge \mathrm{d}x \wedge \mathrm{d}z + \mathrm{d}\left(\cos(t-x)\right) \wedge \mathrm{d}x \wedge \mathrm{d}y \\
&= -\cos(t-x)\left(\mathrm{d}t - \mathrm{d}x\right) \wedge \mathrm{d}x \wedge \mathrm{d}z - \sin(t-x)\left(\mathrm{d}t - \mathrm{d}x\right) \wedge \mathrm{d}x \wedge \mathrm{d}y \\
&= -\cos(t-x)\mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}z - \sin(t-x)\mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y.
\end{aligned}$$

$$\star B = -\sin(t-x)\mathrm{d}t \wedge \mathrm{d}y - \cos(t-x)\mathrm{d}t \wedge \mathrm{d}z,$$

since

$$\mathrm{d}x \wedge \mathrm{d}z \wedge (\mathrm{d}t \wedge \mathrm{d}y) = -\mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z)$$

and

$$\mathrm{d}x \wedge \mathrm{d}y \wedge (\mathrm{d}t \wedge \mathrm{d}z) = \mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.$$

Exercise 5.5. For vacuum Maxwell's equations, the transformation

$$\begin{aligned}
\mathbf{B} &\mapsto \mathbf{E} \\
\mathbf{E} &\mapsto -\mathbf{B}
\end{aligned}$$

takes the first pair of equations to the second pair and vice versa!

$$\begin{aligned}
&\nabla \cdot \mathbf{B} \qquad && \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \\
&\nabla \cdot \mathbf{E} \qquad && \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = 0
\end{aligned} \tag{5.18}$$

This symmetry is a duality and is a clue that the electric and magnetic fields are part of a unified whole: the electromagnetic field.

(a) Show that from the perspective of F, this duality amounts to

$$F \mapsto \star F \tag{5.19}$$

(b) Show that if F solves the vacuum Maxwell's equations, then $\star F$ solves the vacuum Maxwell's equations. (*Note:* This symmetry between $\mathbf{E}$ and $\mathbf{B}$ does not, however, extend to the non-vacuum Maxwell equations.)

(c) Show that the complex-valued solution

$$\mathbf{E} = \begin{pmatrix} 0 \\ \mathrm{e}^{\mathrm{i}(t-x)} \\ -\mathrm{i}\mathrm{e}^{\mathrm{i}(t-x)} \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 0 \\ -\mathrm{i}\mathrm{e}^{\mathrm{i}(t-x)} \\ -\mathrm{e}^{\mathrm{i}(t-x)} \end{pmatrix} \tag{5.20}$$

is a self-dual solution of Maxwell's equations on $\mathbb{R}^{3,1}$, i.e. the corresponding electromagnetic field F satisfies $\star F = \mathrm{i}F$. (Since we're using the Lorentzian metric, $\star^2 = -1$, hence F is self-dual if $\star F = \mathrm{i}F$. In contrast, in the Euclidean metric on $\mathbb{R}^4$, F is self-dual if $\star F = F$.)

Argue that the real part (and similarly the imaginary part) of $\mathbf{E}$ and $\mathbf{B}$

$$\mathbf{E}_{\mathrm{Re}} = \begin{pmatrix} 0 \\ \cos(t-x) \\ \sin(t-x) \end{pmatrix} \qquad \mathbf{B}_{\mathrm{Re}} = \begin{pmatrix} 0 \\ \sin(t-x) \\ \cos(t-x) \end{pmatrix} \tag{5.21}$$

give a solution of Maxwell's equations.

Solution: (a) The transformation

$$\mathbf{B} \mapsto \mathbf{E}$$

and

$$\mathbf{E} \mapsto -\mathbf{B}$$

amounts to

$$B \mapsto \star_{\mathbb{R}^3} E \qquad E \mapsto -\star_{\mathbb{R}^3} B$$

in terms of 1 forms and 2-forms on spacetimes $S \times \mathbb{R}_t$. The relation between $\star_{\mathbb{R}^3}$ and $\star_{\mathbb{R}^{3,1}}$ is given in the remark above: $\star_{\mathbb{R}^3} E = \star_{\mathbb{R}^{3,1}}(E \wedge \mathrm{d}t)$, e.g.

$$\star_{\mathbb{R}^3} \mathrm{d}x = \mathrm{d}y \wedge \mathrm{d}z \qquad \star_{\mathbb{R}^{3,1}} (\mathrm{d}x \wedge \mathrm{d}t) = \mathrm{d}y \wedge \mathrm{d}z;$$

and likewise, $(-\star_{\mathbb{R}^3} B) \wedge \mathrm{d}t = \star_{\mathbb{R}^{3,1}} B$, e.g.

$$-(\star_{\mathbb{R}^3} \mathrm{d}x \wedge \mathrm{d}y) \wedge \mathrm{d}t = -\mathrm{d}z \wedge \mathrm{d}t \qquad \star_{\mathbb{R}^{3,1}} \mathrm{d}x \wedge \mathrm{d}y = -\mathrm{d}z \wedge \mathrm{d}t,$$

The electromagnetic field is

$$F = B + E \wedge \mathrm{d}t.$$

Consequently, making the above transformation gives

$$B \mapsto \star_{\mathbb{R}^{3,1}}(E \wedge \mathrm{d}t) \qquad E \wedge \mathrm{d}t \mapsto \star_{\mathbb{R}^{3,1}} B.$$

Hence,

$$F = B + E \wedge \mathrm{d}t \mapsto \star_{\mathbb{R}^{3,1}}(F) = \star_{\mathbb{R}^{3,1}}(B + E \wedge \mathrm{d}t).$$

(b) If F solves $\star\mathrm{d}\star F = 0$ and $\mathrm{d}F = 0$, then clearly $\star F$ solves $\star\mathrm{d}\star F$ and $\mathrm{d}F = 0$.

(c) We compute

$$\begin{aligned}
\star F &= \star\left(-\mathrm{i}e^{\mathrm{i}(t-x)}\mathrm{d}z \wedge \mathrm{d}x - e^{\mathrm{i}(t-x)}\mathrm{d}x \wedge \mathrm{d}y + e^{\mathrm{i}(t-x)}\mathrm{d}y \wedge \mathrm{d}t - \mathrm{i}e^{\mathrm{i}(t-x)}\mathrm{d}z \wedge \mathrm{d}t\right) \\
&= -\mathrm{i}e^{\mathrm{i}(t-x)}\mathrm{d}t \wedge \mathrm{d}y - e^{\mathrm{i}(t-x)}\mathrm{d}t \wedge \mathrm{d}z + e^{\mathrm{i}(t-x)}\mathrm{d}z \wedge \mathrm{d}x - \mathrm{i}e^{\mathrm{i}(t-x)}\mathrm{d}x \wedge \mathrm{d}y \\
&= \mathrm{i}F.
\end{aligned}$$

If self-dual F solves Maxwell's equations (or even just $\mathrm{d}F = 0$, it follows that the real and imaginary parts are separately solution of Maxwell's equations, since $\star, \mathrm{d}$ do not mix the real and imaginary parts. Moreover, note that $\star\mathrm{Re}(F) = -\mathrm{Im}(F)$. Taking the real part, we see that the associated electromagnetic potential is

$$\mathrm{Re}F = \underbrace{\sin(t-x)\mathrm{d}z \wedge \mathrm{d}x - \cos(t-x)\mathrm{d}x \wedge \mathrm{d}y}_{B} + \left(\underbrace{\cos(t-x)\mathrm{d}y + \sin(t-x)\mathrm{d}z}_{E}\right) \wedge \mathrm{d}t. \tag{5.22}$$

We compute:

$$\begin{aligned}
\mathrm{d}\ \mathrm{Re}F &= \cos(t-x)\mathrm{d}t \wedge \mathrm{d}z \wedge \mathrm{d}x + \sin(t-x)\mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y \\
&\quad + \sin(t-x)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}y - \cos(t-x)\mathrm{d}x \wedge \mathrm{d}z \wedge \mathrm{d}t \\
&= 0
\end{aligned}$$

Similarly, we see that

$$\begin{aligned}
\star\mathrm{Re}F &= \star\left(\sin(t-x)\mathrm{d}z \wedge \mathrm{d}x - \cos(t-x)\mathrm{d}x \wedge \mathrm{d}y + \cos(t-x)\mathrm{d}y \wedge \mathrm{d}t + \sin(t-x)\mathrm{d}z \wedge \mathrm{d}t\right) \\
&= \sin(t-x)\mathrm{d}t \wedge \mathrm{d}y - \cos(t-x)\mathrm{d}t \wedge \mathrm{d}z + \cos(t-x)\mathrm{d}x \wedge \mathrm{d}z - \sin(t-x)\mathrm{d}x \wedge \mathrm{d}y
\end{aligned}$$

hence, $\mathrm{d}\ \star\mathrm{Re}F = 0$.

Exercise 5.6. For notational convenience let $x_0 = t, x_1 = x, x_2 = y, x_3 = z$. Define

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} = A \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}, \tag{5.23}$$

where $A \in SO(3,1)$, i.e. take an arbitrary Lorentzian transformation and a translation. Compute

$$-(\mathrm{d}y_0)^2 + (\mathrm{d}y_1)^2 + (\mathrm{d}y_2)^2 + (\mathrm{d}y_3)^2,$$

and show that it's equal to

$$g = -(\mathrm{d}x_0)^2 + (\mathrm{d}x_1)^2 + (\mathrm{d}x_2)^2 + (\mathrm{d}x_3)^2,$$

i.e. the coordinate transformation above is an isometry of $\mathbb{R}^{3,1}$ to itself.

Solution:

$$\begin{aligned}
& -(\mathrm{d}y_0)^2 + (\mathrm{d}y_1)^2 + (\mathrm{d}y_2)^2 + (\mathrm{d}y_3)^2 \\
= \quad & -\left(\sum_{i=0}^{3} \frac{\partial y_0}{\partial x_i}\mathrm{d}x_i\right)^2 + \left(\sum_{i=0}^{3} \frac{\partial y_1}{\partial x_i}\mathrm{d}x_i\right)^2 + \left(\sum_{i=0}^{3} \frac{\partial y_2}{\partial x_i}\mathrm{d}x_i\right)^2 + \left(\sum_{i=0}^{3} \frac{\partial y_3}{\partial x_i}\mathrm{d}x_i\right)^2 \\
= \quad & -\left(\sum_{i=0}^{3} A_{0,i}\mathrm{d}x_i\right)^2 + \left(\sum_{i=0}^{3} A_{1,i}\mathrm{d}x_i\right)^2 + \left(\sum_{i=0}^{3} A_{2,i}\mathrm{d}x_i\right)^2 + \left(\sum_{i=0}^{3} A_{3,i}\mathrm{d}x_i\right)^2 \\
= \quad & \sum_{0\le i<j\le 3} \left(-2A_{0,i}A_{0,j} + 2A_{1,i}A_{1,j} + 2A_{2,i}A_{2,j} + 2A_{3,i}A_{3,j}\right)\mathrm{d}x_i\mathrm{d}x_j \\
& + \sum_{i=0}^{3}\left(-A_{0,i}^2 + A_{1,i}^2 + A_{2,i}^2 + A_{3,i}^2\right)\mathrm{d}x_i^2 \\
= \quad & \sum_{0\le i<j\le 3} \left(2A^TGA\right)_{ij}\mathrm{d}x_i\mathrm{d}x_j + \sum_{i=0}^{3}\left(A^TGA\right)_{ii}\mathrm{d}x_i^2 \\
= \quad & \sum_{0\le i<j\le 3} (2G)_{ij}\,\mathrm{d}x_i\mathrm{d}x_j + \sum_{i=0}^{3}(G)_{ii}\,\mathrm{d}x_i^2 \\
= \quad & -\mathrm{d}x_0^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2.
\end{aligned}$$

Exercise 5.7. [Bonus] We say we are in temporal gauge if the potential $\widetilde{A}$ satisfies $\widetilde{A}\left(\frac{\partial}{\partial t}\right) = 0$, i.e.

$$\widetilde{A} = 0\ \mathrm{d}t + \widetilde{A}_x\mathrm{d}x + \widetilde{A}_y\mathrm{d}y + \widetilde{A}_z\mathrm{d}z. \tag{5.24}$$

Given an arbitrary potential $\widetilde{A}$, define a function f such that $\widetilde{A}' = \widetilde{A} - \mathrm{d}f$ is in temporal gauge.

Solution: Take

$$f(t,p) = \int_0^t \widetilde{A}_t(s,p)\mathrm{d}s. \tag{5.25}$$

Then

$$\widetilde{A}_t'(t,p) = \widetilde{A}_t(t,p) - (\mathrm{d}f)(\frac{\partial}{\partial t})(t,p) = A_0(t,p) - \partial_t\int_0^t A_0(s,p)\mathrm{d}s = 0. \tag{5.26}$$

Find f such that$\widetilde{A}' = \widetilde{A} - \mathrm{d}f.$ is in temporal gauge.

6. Lecture 6: Action Principle; Lie Groups and Lie Algebras (LF)

[(given Tuesday, July 9, 2019)]

Exercises.

Exercise 6.1. [Straightforward] A path bewetween two points $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ is a map

$$\gamma : [0,1] \to \mathbb{R}^n, \tag{6.1}$$

such that $\gamma(0) = \mathbf{a}$ and $\gamma(1) = \mathbf{b}$. The length of the path is

$$S(\gamma) = \int_0^1 \sqrt{\gamma'(t) \cdot \gamma'(t)} \mathrm{d}t. \tag{6.2}$$

Find the Euler-Lagrange equations for the action S.

Without loss of generality, we can assume that the path γ is traced out at "constant speed" $\sqrt{\gamma'(t) \cdot \gamma'(t)} = c$ (hence $\gamma'(t) \cdot \gamma''(t) = 0$). Using this, show that a length-minimizing path must be a straight line.

[Bonus] Show that a length-minimizing path must be a straight line without assuming that γ is traced out at constant speed.

Solution: A deformation is $\delta\gamma : [0,1] \to \mathbb{R}^n$ such that $\delta\gamma(0) = 0$ and $\delta\gamma(1) = 0$. We then see that

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} S(\gamma + \epsilon\delta\gamma) &= \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \int_0^1 \sqrt{(\gamma'(t) + \epsilon\delta\gamma'(t)) \cdot (\gamma'(t) + \epsilon\delta\gamma'(t))} \mathrm{d}t \\
&= \int_0^1 \frac{1}{2} (\gamma'(t) \cdot \gamma'(t))^{-1/2} \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} (\gamma'(t) + \epsilon\delta\gamma'(t)) \cdot (\gamma'(t) + \epsilon\delta\gamma'(t)) \mathrm{d}t \\
&= \int_0^1 \frac{1}{2} (\gamma'(t) \cdot \gamma'(t))^{-1/2} \, 2\gamma'(t) \cdot \delta\gamma'(t) \mathrm{d}t \\
&= -\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\gamma'(t)}{\sqrt{\gamma'(t) \cdot \gamma'(t)}} \right) \cdot \delta\gamma(t)) \mathrm{d}t,
\end{aligned}$$

Hence the Euler-Lagrange equations are

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\gamma'(t)}{\sqrt{\gamma'(t) \cdot \gamma'(t)}} \right) \\
&= \frac{(\gamma'(t) \cdot \gamma'(t))^{1/2} \gamma''(t) - \gamma'(t) (\gamma'(t) \cdot \gamma'(t))^{-1/2} (\gamma'(t) \cdot \gamma''(t))}{\gamma'(t) \cdot \gamma'(t)} \\
&= \frac{(\gamma'(t) \cdot \gamma'(t)) \gamma''(t) - \gamma'(t) (\gamma'(t) \cdot \gamma''(t))}{(\gamma'(t) \cdot \gamma'(t))^{3/2}}
\end{aligned}$$

It follows that

$$0 = \left(\gamma'(t) \cdot \gamma'(t)\right) \gamma''(t) - \gamma'(t) \left(\gamma'(t) \cdot \gamma''(t)\right). \tag{6.3}$$

Now supposing $\|\gamma'(t)\| = c$, then the Euler-Lagrange equations become

$$\gamma''(t) = 0, \tag{6.4}$$

i.e. $\gamma(t) = \mathbf{c}_1 + \mathbf{c}_2 t$. This is the equation for a straight line. The vectors $\mathbf{c}_1$ and $\mathbf{c}_2$ are determined from $\mathbf{a}$ and $\mathbf{b}$.

[Bonus] From the Euler-Lagrange equations we have

$$0 = \left(\gamma'(t) \cdot \gamma'(t)\right) \gamma''(t) - \gamma'(t) \left(\gamma'(t) \cdot \gamma''(t)\right). \tag{6.5}$$

Taking the dot product of this with $\gamma''(t)$, we see that

$$\|\gamma'(t)\|^2 \|\gamma''(t)\|^2 = \langle \gamma'(t), \gamma''(t) \rangle^2. \tag{6.6}$$

It follows from the Cauchy-Schwarz inequality that $\gamma''(t)$ and $\gamma'(t)$ are linearly dependent. Since $\gamma''(t)$ always points in the same direction as $\gamma'(t)$, we see that (provided $\gamma'(t)$ never vanishes) $\gamma(t)$ is a straight line.

Exercise 6.2. Let $f : [a, b] \to \mathbb{R}^{\geq 0}$. Consider the surface of revolution obtained by rotating the graph of f around the x axis. The surface area of the obtained surface is

$$S(f) = \int_a^b 2\pi f(x) \sqrt{1 + f'(x)^2} \mathrm{d}x. \tag{6.7}$$

Show that the Euler-Lagrange equations are

$$\sqrt{1 + f'(x)^2} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{f(x) f'(x)}{\sqrt{1 + f'(x)^2}} \right) = 0, \tag{6.8}$$

i.e. $f(x)$ solves the differential equation

$$1 + (f'(x))^2 = f(x) f''(x). \tag{6.9}$$

[Bonus] Solve this ODE for $f(x)$. [You should find $f(x) = \cosh\left(\frac{x + c_1}{c_2}\right)$, where c_1, c_2 are arbitrary constants.] This resulting surface of rotation is called a minimal surface.

<u>Solution:</u> We take

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} S(f + \epsilon\delta f) &= \int_a^b 2\pi (f(x) + \epsilon \delta f(x)) \sqrt{1 + (f'(x) + \epsilon \delta f'(x))^2} \mathrm{d}x \\
&= 2\pi \int_a^b \left(\delta f(x) \sqrt{1 + f'(x)^2} + f(x) \left(1 + f'(x)^2\right)^{-1/2} f'(x) \delta f'(x) \right) \mathrm{d}x \\
&= 2\pi \int_a^b \delta f(x) \left(\sqrt{1 + f'(x)^2} - \frac{\mathrm{d}}{\mathrm{d}x} \left(f(x) \left(1 + f'(x)^2\right)^{-1/2} f'(x) \right) \right) \mathrm{d}x,
\end{aligned}$$

where in the last line we used integration by parts. Thus, we see that the Euler-Lagrange equations are as claimed.

Expanding this, we see

$$\begin{aligned}
0 &= \sqrt{1+f'(x)^2} - \frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)f'(x)\left(1+f'(x)^2\right)^{-1/2}\right) \\
&= \sqrt{1+f'(x)^2} - \left(f'(x)f'(x)\left(1+f'(x)^2\right)^{-1/2}\right) - \left(f(x)f''(x)\left(1+f'(x)^2\right)^{-1/2}\right) \\
&\quad + \frac{1}{2}\left(f(x)f'(x)\left(1+f'(x)^2\right)^{-3/2} 2f'(x)f''(x)\right)
\end{aligned}$$

Multiplying by $\sqrt{1+f'(x)^2}$, we get

$$\begin{aligned}
0 &= 1+f'(x)^2 - f'(x)f'(x) - f(x)f''(x) + f(x)f'(x)f'(x)f''(x)(1+f'(x)^2)^{-1} \\
&= 1 - f(x)f''(x) + f(x)f'(x)f'(x)f''(x)(1+f'(x)^2)^{-1} \\
&= 1 + f(x)f''(x)\left(-\frac{1+f'(x)^2}{1+f'(x)^2} + \frac{f'(x)^2}{1+f'(x)^2}\right) \\
&= 1 - \frac{f(x)f''(x)}{1+f'(x)^2}
\end{aligned}$$

Hence, we indeed see that

$$1+f'(x)^2 = f(x)f''(x). \tag{6.10}$$

[Bonus] This is a difficult ODE to solve. One technique is to first differentiate both sides. In this, we seemingly make it worse but get rid of the pesky "1".

$$\begin{aligned}
1+f'(x)^2 &= f(x)f''(x) \\
\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\left(1+f'(x)^2\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)f''(x)\right) \\
\Rightarrow \quad 2f'(x)f''(x) &= f(x)f'''(x) + f'(x)f''(x) \\
\Rightarrow \quad 0 &= f(x)f'''(x) - f'(x)f''(x) \\
\Rightarrow \quad 0 &= \frac{f(x)f'''(x) - f'(x)f''(x)}{f(x)^2} \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f''(x)}{f(x)}\right) \\
\Rightarrow \quad c^2 &= \frac{f''(x)}{f(x)} \\
\Rightarrow \quad c^2 f(x) &= f''(x) \\
\Rightarrow \quad f(x) &= c_1 \sinh(cx) + c_2 \cosh(cx)
\end{aligned}$$

Now, there are three constants c, c_1, c_2 rather than two because we took a derivative. Plugging this in to our original ODE, we see that the constants must satisfy

$$1 + c^2(c_1 - c_2)(c_1 + c_2) = 0. \tag{6.11}$$

We let $c_1 = \sinh(\phi)/c$ and $c_2 = \cosh(\phi)/c$, and then see that this equation is satisfied because $\sinh^2(\phi) - \cosh^(\phi) = -1$. Then we see that

$$f(x) = \frac{1}{c}\left(\sinh(\phi)\sinh(cx) + \cosh(\phi)\cosh(cx)\right) = \frac{1}{c}\cosh(\phi + cx), \tag{6.12}$$

for constants ϕ and c.

Exercise 6.3. Show that for any diagonalizable matrix B

$$\det(\exp(B)) = \exp(\mathrm{tr}B).$$

Note that because diagonalizable matrices are dense in the space of all matrices it follows that the above equation is true for *any* matrix.

Use this to show that the Lie algebra $\mathfrak{sl}(n, \mathbb{C})$ of $SL(n, \mathbb{C})$ consists of all $n \times n$ traceless complex matrices, while the Lie algebra $\mathfrak{sl}(n, \mathbb{R})$ consists of all $n \times n$ traceless real matrices.

Solution: If $D = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$ is a diagonal matrix then

$$e^D = \mathrm{diag}(e^{\lambda_1}, \cdots, e^{\lambda_n}) \tag{6.13}$$

Hence,

$$\det(\exp(D)) = \prod_{i=1}^{n} \exp(\lambda_i) = \exp(\sum_{i=1}^{n} \lambda_i) = \exp(\mathrm{tr}(D)) \tag{6.14}$$

If $B = g^{-1}Dg$ for D a diagonal matrix, then

$$\exp(B) = g^{-1}\exp(D)g,$$

hence

$$\begin{aligned}
\det(\exp(B)) &= \det(g^{-1})\det(\exp(D))\det(g) \\
&= \det(\exp(D)) \\
&= \exp(\mathrm{tr}(D)) \\
&= \exp(\mathrm{tr}(g^{-1}Dg)) \\
&= \exp(\mathrm{tr}(B)).
\end{aligned}$$

Now, if $\det(I + \epsilon B) = 1$ to first order in ϵ, then we see, that to first order

$$\begin{aligned}
\det(I + \epsilon B) &= \det(\exp(\epsilon B)) + O(\epsilon^2) \\
&= \exp(\epsilon \mathrm{tr}(B)) + O(\epsilon^2) \\
&= 1 + \epsilon \mathrm{tr}(B) + O(\epsilon^2).
\end{aligned}$$

Hence,

$$\mathfrak{sl}(n, \mathbb{R}) = \{B \in \mathrm{Mat}_{n\times n}(\mathbb{R}) : \mathrm{tr}(B) = 0\} \tag{6.15}$$

and similarly

$$\mathfrak{sl}(n, \mathbb{C}) = \{B \in \mathrm{Mat}_{n\times n}(\mathbb{C}) : \mathrm{tr}(B) = 0\}. \tag{6.16}$$