# The asymptotic geometry of the Hitchin moduli space

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# The Hitchin moduli space

Fixed data:

- $C$, a compact Riemann surface
- $G = SU(n)$, $G_{\mathbb{C}} = SL(n, \mathbb{C})$
- $E \to C$, a complex vector bundle of rank $n$ with $\mathrm{Aut}(E) = SL(E)$

$\rightsquigarrow$ Hitchin moduli space, $\mathcal{M}$.

Fact #1: $\mathcal{M}$ is a noncompact hyperkähler manifold with metric $g_{L^2}$
$\Rightarrow$ have a $\mathbb{CP}^1$-family of Kähler manifolds $\mathcal{M}_\zeta = (\mathcal{M}, g_{L^2}, I_\zeta, \omega_\zeta)$.

- $\mathcal{M}_{\zeta=0}$ is $G_{\mathbb{C}}$-Higgs bundle moduli space
- $\mathcal{M}_{\zeta \in \mathbb{C}^\times}$ is moduli space of flat $G_{\mathbb{C}}$-connections

# The Higgs bundle moduli space

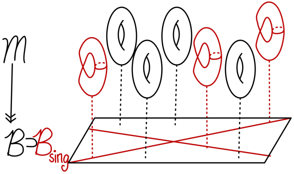
**Definition**

A **Higgs bundle** is a pair $(\bar{\partial}_E, \varphi)$ consisting of a holomorphic structure $\bar{\partial}_E$ on $E$ and a "Higgs field" $\varphi \in \Omega^{1,0}(C, \mathrm{End}_0 E)$ such that $\bar{\partial}_E \varphi = 0$.

(Locally, $\bar{\partial}_E = \bar{\partial}$ and $\varphi = P \mathrm{d}z$, where $P$ is a tracefree $n \times n$ matrix with holomorphic entries.)

**Ex:** The $GL(1)$-Higgs bundle moduli space is $\mathcal{M} = \underbrace{\mathrm{Jac}(C)}_{\bar{\partial}_E} \times \underbrace{H^0(K_C)}_{\varphi}$.
For $C = T^2_\tau$, $\mathcal{M} = T^2_\tau \times \mathbb{C}$

Fact #2: In its avatar as Higgs bundle moduli space, $\mathcal{M}$ is an algebraic completely integrable system.



# Hitchin's equations

Hitchin's equations are equations for a hermitian metric $h$ on $E$.

**Definition**

A Higgs bundle $(\bar{\partial}_E, \varphi)$, together with a Hermitian metric $h$ on $E$, is a **solution of Hitchin's equations** if

$$F_D^\perp + [\varphi, \varphi^{*_h}] = 0.$$

(Here, $D$ is the Chern connection for $(\bar{\partial}_E, h)$.)

There is a correspondence between stable Higgs bundles and solutions of Hitchin's equations. [Hitchin, Simpson]

$$\left\{ \begin{matrix} \text{stable Higgs bundles} \\ (\bar{\partial}_E, \varphi) \end{matrix} \right\} \Big/ \mathcal{SL}(E) \xleftrightarrow{\cong} \left\{ \begin{matrix} \text{soln of Hitchin's eqn} \\ (\bar{\partial}_E, \varphi, h) \end{matrix} \right\} \Big/ \mathcal{SU}(E) =: \mathcal{M}$$

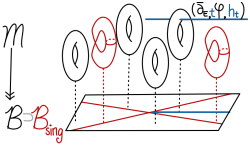
# Two hyperkähler metrics on the regular locus $\mathcal{M}'$

- $g_{L^2}$ Hitchin's $L^2$ hyperkähler metric—uses $h$
- $g_{\mathrm{sf}}$ semiflat metric—from integrable system structure

**Gaiotto-Moore-Neitzke's Conjecture**

Fix $(\bar\partial_E, \varphi) \in \mathcal{M}'$. Along the ray $T_{(\bar\partial_E, t\varphi, h_t)}\mathcal{M}'$,

$$g_{L^2} - g_{\mathrm{sf}} = \Omega \mathrm{e}^{-\ell t} + \text{faster decaying}$$




Progress:

- Mazzeo-Swoboda-Weiss-Witt proved polynomial decay for $SU(2)$-Hitchin moduli space. ['17]
- Dumas-Neitzke proved exponential decay in $SU(2)$-Hitchin section with its tangent space. ['18]
- **F** proved exponential* decay for $SU(n)$-Hitchin moduli space. ['18]
- **F**-Mazzeo-Swoboda-Weiss proved exponential* decay for $SU(2)$ parabolic Hitchin moduli space. (Higgs field has simple poles along divisor $D \subset C$.) ['19]

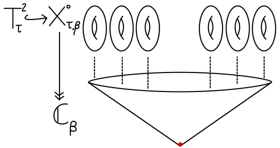
*: Rate of exponential decay is not optimal.

# Analogy from noncompact hyperkähler four-manifolds $X$

Categories based on asymptotic volume growth: ALE/ALF/ALG/ALH

ALE: Any $X$ is asymptotic to some standard model $X_\Gamma^\circ = \mathbb{C}^2/\Gamma$ where $\Gamma$ is a finite subgroup of $SU(2)$. [Kronheimer]

ALG: Any $X$ (with faster than quadratic curvature decay) is asymptotic to some standard model $X_{\tau,\beta}^\circ$ fibered over $\mathbb{C}_\beta$ of angle $2\pi\beta$ with fiber $T_\tau^2$. [Chen-Chen]




## Proposition [F-Mazzeo-Swoboda-Weiss]

The moduli space of strongly parabolic $SL(2,\mathbb{C})$-Higgs bundles on the four-punctured sphere is an ALG gravitational instanton. In this case, $g_{\mathrm{sf}}$ is the standard model metric of Chen-Chen.

Hitchin moduli spaces are expected to be QALG. Roughly,

$$g_{\mathrm{sf}} \sim \text{standard model metric}$$

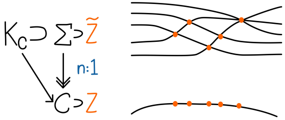
# Main Theorem

**Theorem [F, F-Mazzeo-Swoboda-Weiss]**

Fix $(\bar\partial_E, \varphi) \in \mathcal{M}'$ and a Higgs bundle variation $(\dot\eta, \dot\varphi) \in T_{(\bar\partial_E,\varphi)}\mathcal{M}$.
Along the ray $T_{(\bar\partial_E, t\varphi, h_t)}\mathcal{M}'$, as $t \to \infty$,

$$\|(\dot\eta, t\dot\varphi)\|^2_{g_{L^2}} - \|(\dot\eta, t\dot\varphi)\|^2_{g_{\mathrm{sf}}} = O(\mathrm{e}^{-\varepsilon t})$$

As $t \to \infty$, $F_{D(\bar\partial_E, h_t)}$ concentrates along branch divisor $Z \subset C$.
The limiting metric $h_\infty$ is flat with singularities along $Z$.




The main difficulty is dealing with the contributions to the integral from infinitesimal neighborhoods around $Z$.

# Idea #1: Semiflat metric is an $L^2$-metric

Hitchin's hyperkähler metric $g_{L^2}$ on $T_{(\bar{\partial}_E, t\varphi)}\mathcal{M}$ is

$$\|(\dot{\eta}, t\dot{\varphi}, \dot{\nu}_t)\|^2_{g_{L^2}} = 2\int_C \left|\dot{\eta} - \bar{\partial}_E \dot{\nu}_t\right|^2_{h_t} + t^2 \left|\dot{\varphi} + [\dot{\nu}_t, \varphi]\right|^2_{h_t}$$

where the metric variation $\dot{\nu}_t$ of $h_t$ is the unique solution of

$$\partial_E^{h_t} \bar{\partial}_E \dot{\nu}_t - \partial_E^h \dot{\eta} - t^2 \left[\varphi^{*_{h_t}}, \dot{\varphi} + [\dot{\nu}_t, \varphi]\right] = 0.$$

The semiflat metric, from the integrable system structure, on $T_{(\bar{\partial}_E, t\varphi)}\mathcal{M}$ is an $L^2$-metric defined using $h_\infty$.

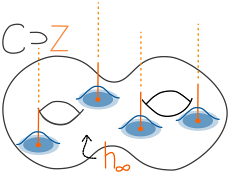
$$\|(\dot{\eta}, t\dot{\varphi}, \dot{\nu}_\infty)\|^2_{g_{\mathrm{sf}}} = 2\int_C \left|\dot{\eta} - \bar{\partial}_E \dot{\nu}_\infty\right|^2_{h_\infty} + t^2 \left|\dot{\varphi} + [\dot{\nu}_\infty, \varphi]\right|^2_{h_\infty},$$

where the metric variation $\dot{\nu}_\infty$ of $h_\infty$ is independent of $t$ and solves

$$\partial_E^{h_t} \bar{\partial}_E \dot{\nu}_\infty - \partial_E^h \dot{\eta} = 0 \qquad \left[\varphi^{*_{h_\infty}}, \dot{\varphi} + [\dot{\nu}_\infty, \varphi]\right] = 0.$$

# Idea #2: Approximate solutions

Desingularize $h_\infty$ (singular at $Z$) by gluing in solutions $h_t^{\mathrm{model}}$ of Hitchin's equations on neighborhoods of $p \in Z$. $\rightsquigarrow h_t^{\mathrm{approx}}$.




Perturb $h_t^{\mathrm{approx}}$ to an actual solution $h_t$ using a contracting mapping argument.

*(Difficulty: Showing the first eigenvalue of $L_t : H^2 \to L^2$ is $\geq Ct^{-2}$ )*

**Theorem**

$$h_t(v, w) = h_t^{\mathrm{app}}(\mathrm{e}^{\gamma_t} v, \mathrm{e}^{\gamma_t} w) \qquad \text{for } \|\gamma_t\|_{H^2} \leq \mathrm{e}^{-\varepsilon t}.$$

# Idea #2: Approximate solutions

Define an $L^2$-metric *non-hyperkähler metric* $g_{\mathrm{app}}$ on $\mathcal{M}'$ using variations of the metric $h_t^{\mathrm{app}}$.

$$\begin{aligned}
\|(\dot\eta, t\dot\varphi, \dot\nu_t)\|^2_{g_{L^2}} &= 2\int_C \left|\dot\eta - \bar\partial_E \dot\nu_t\right|^2_{h_t} + t^2\left|\dot\varphi + [\dot\nu_t, \varphi]\right|^2_{h_t} \\
\|(\dot\eta, t\dot\varphi, \dot\nu_\infty)\|^2_{g_{\mathrm{sf}}} &= 2\int_C \left|\dot\eta - \bar\partial_E \dot\nu_\infty\right|^2_{h_\infty} + t^2\left|\dot\varphi + [\dot\nu_\infty, \varphi]\right|^2_{h_\infty} \\
\|(\dot\eta, t\dot\varphi, \dot\nu_t^{\mathrm{app}})\|^2_{g_{\mathrm{app}}} &= 2\int_C \left|\dot\eta - \bar\partial_E \dot\nu_t^{\mathrm{app}}\right|^2_{h_t^{\mathrm{app}}} + t^2\left|\dot\varphi + [\dot\nu_t^{\mathrm{app}}, \varphi]\right|^2_{h_t^{\mathrm{app}}},
\end{aligned}$$

Then, break the $g_{L^2} - g_{\mathrm{sf}}$ into two piece:

$$\left(\|(\dot\eta, t\dot\varphi, \dot\nu_t)\|^2_{g_{L^2}} - \|(\dot\eta, t\dot\varphi, \dot\nu_t^{\mathrm{app}})\|^2_{g_{\mathrm{app}}}\right) + \left(\|(\dot\eta, t\dot\varphi, \dot\nu_t^{\mathrm{app}})\|^2_{g_{\mathrm{app}}} - \|(\dot\eta, t\dot\varphi, \dot\nu_\infty)\|^2_{g_{\mathrm{sf}}}\right)$$

## Corollary

Since $h_t(v, w) = h_t^{\mathrm{app}}(\mathrm{e}^{\gamma_t} v, \mathrm{e}^{\gamma_t} w)$ for $\|\gamma_t\|_{H^2} \leq \mathrm{e}^{-\varepsilon t}$, along the ray $T_{(\bar\partial_E, t\varphi)}\mathcal{M}$, as $t \to \infty$,

$$\|(\dot\eta, t\dot\varphi, \dot\nu_t)\|^2_{g_{L^2}} - \|(\dot\eta, t\dot\varphi, \dot\nu_t^{\mathrm{app}})\|^2_{g_{\mathrm{app}}} = O(\mathrm{e}^{-\varepsilon t})$$

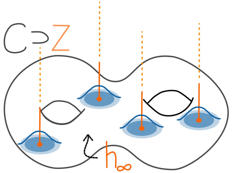
# Idea #2: Approximate solutions

Our goal is to show that the following sum is $O(\mathrm{e}^{-\varepsilon t})$:

$$\underbrace{\left(\|(\dot\eta, t\dot\varphi, \dot\nu_t)\|^2_{g_{L^2}} - \|(\dot\eta, t\dot\varphi, \dot\nu_t^{\mathrm{app}})\|^2_{g_{\mathrm{app}}}\right)}_{O(\mathrm{e}^{-\varepsilon t})} + \left(\|(\dot\eta, t\dot\varphi, \dot\nu_t^{\mathrm{app}})\|^2_{g_{\mathrm{app}}} - \|(\dot\eta, t\dot\varphi, \dot\nu_\infty)\|^2_{g_{\mathrm{sf}}}\right)$$

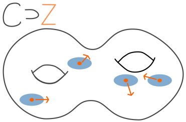
It remains to show that $\|(\dot\eta, t\dot\varphi, \dot\nu_t^{\mathrm{app}})\|^2_{g_{\mathrm{app}}} - \|(\dot\eta, t\dot\varphi, \dot\nu_\infty)\|^2_{g_{\mathrm{sf}}} = O(\mathrm{e}^{-\varepsilon t})$.

Since $h_t^{\mathrm{app}}$ differs from $h_\infty$ only on disks around $p \in Z$, the difference $g_{\mathrm{app}} - g_{\mathrm{sf}}$ localizes (up to exponentially decaying errors) to disks around $p \in Z$.



# Idea #3: Holomorphic variations

When Mazzeo-Swoboda-Weiss-Witt proved that $g_{L^2} - g_{\mathrm{sf}}$ was at least polynomially decaying in $t$, all of their possible polynomial terms came from infinitesimal variations in which the branch points move.




Dumas-Neitzke use a family of biholomorphic maps on local disks (originally defined by Hubbard-Masur) to match the changing location of the branch points. This uses subtle geometry of Hitchin moduli space. E.g. for $SU(2)$, conformal invariance.

Remarkably, this can be generalized off of the Hitchin section and from $SU(2)$ to $SU(n)$.

## Theorem

$$\|(\dot\eta, t\dot\varphi, \dot\nu_t^{\mathrm{app}})\|^2_{g_{\mathrm{app}}} - \|(\dot\eta, t\dot\varphi, \dot\nu_\infty)\|^2_{g_{\mathrm{sf}}} = O(\mathrm{e}^{-\varepsilon t})$$

# Main Theorem

## Gaiotto-Moore-Neitzke's Conjecture

Fix $(\bar{\partial}_E, \varphi) \in \mathcal{M}'$. Along the ray $T_{(\bar{\partial}_E, t\varphi, h_t)}\mathcal{M}'$,

$$\|(\dot{\eta}, t\dot{\varphi})\|^2_{g_{L^2}} - \|(\dot{\eta}, t\dot{\varphi})\|^2_{g_{\mathrm{sf}}} = \Omega \mathrm{e}^{-\ell t} + \text{faster decaying}$$

## Theorem [**F**, **F**-Mazzeo-Swoboda-Weiss]

Fix $(\bar{\partial}_E, \varphi) \in \mathcal{M}'$ and a Higgs bundle variation $(\dot{\eta}, \dot{\varphi}) \in T_{(\bar{\partial}_E, \varphi)}\mathcal{M}$. Along the ray $T_{(\bar{\partial}_E, t\varphi, h_t)}\mathcal{M}'$, as $t \to \infty$,

$$\|(\dot{\eta}, t\dot{\varphi})\|^2_{g_{L^2}} - \|(\dot{\eta}, t\dot{\varphi})\|^2_{g_{\mathrm{sf}}} = O(\mathrm{e}^{-\varepsilon t})$$

- **F** proved exponential* decay for $SU(n)$-Hitchin moduli space. ['18]
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*: Rate of exponential decay is not optimal.

Here, $\varepsilon = \frac{\ell}{2} - \delta$ for $\delta$ arbitrarily small.

# Example: The four-punctured sphere

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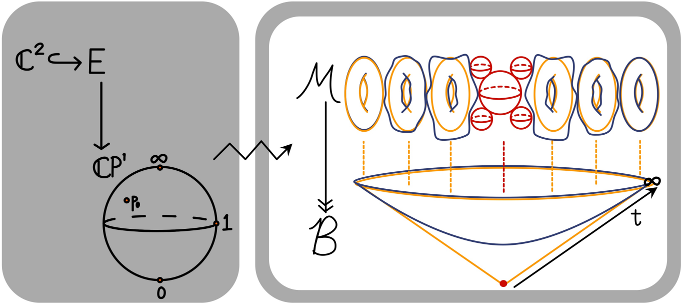
The $SU(2)$-Hitchin moduli space for the four-puncture sphere is of $\dim_{\mathbb{R}} \mathcal{M} = 4$.

**Question**

Is it possible to prove Gaiotto-Moore-Neitzke's conjectured rate of exponential decay in the case of the four-punctured sphere?




In this case, the semiflat metric has cone angle $\pi$ and fibers $T^2_\tau$ where $\tau = \lambda(p_0)$. The volume of the fibers are $2\pi^2$.

# Example: The four-punctured sphere

LeBrun gave a framework to describe all Ricci-flat Kähler metrics of complex-dimension two with a holomorphic circle action in terms of two functions $u, w$.

### Generalized Gibbons-Hawking Ansatz specialized to our case:

Consider a hyperkähler metric on $T^2_{x,y} \times \mathbb{R}^+_r \times S^1_\theta$ with holomorphic circle action. The hyperkähler metric is

$$g_{L^2} = \mathrm{e}^u u_r(\mathrm{d}x^2 + \mathrm{d}y^2) + u_r \mathrm{d}r^2 + u_r^{-1}\mathrm{d}\theta^2$$

where $u : T^2_{x,y} \times \mathbb{R}^+_r \to \mathbb{R}$ solves

$$\Delta_{T^2} u + \partial_r^2 \mathrm{e}^u = 0.$$

The semiflat metric $g_{\mathrm{sf}}$ corresponds to $u_{\mathrm{sf}} = \log r$.

### Goal

Show that $u - u_{\mathrm{sf}}$ has conjectured rate of exponential decay.

# Bootstrapping to optimal exponential decay

Let $v = u - u_{\mathrm{sf}}$. Then,

$$\underbrace{\Delta_T v + r\partial_r^2 v + 2\partial_r v}_{Lv} = \underbrace{-e^v r(\partial_r v)^2 - (e^v - 1)\left(r\partial_r^2 v + 2\partial_r v\right)}_{Q(v,\partial_r v,\partial_{rr} v)},$$

Observation #1: The first exponentially-decaying function in $\ker L$ decays like $\mathrm{e}^{-2\lambda_T\sqrt{r}}$, where $\lambda_T^2$ is the first positive eigenvalue of $-\Delta_{T^2}$. In the torus $T_\tau^2$ with its semiflat metric $\lambda_T^2 = \frac{2}{\mathrm{Im}\ \tau}$.

Observation #2: If $v \sim \mathrm{e}^{-\varepsilon\sqrt{r}}$, then $Q(v, \partial_r v, \partial_{rr} v) \sim \mathrm{e}^{-2\varepsilon\sqrt{r}}$.

Solving the non-homogeneous problem $Lv = f$ for $f \sim \mathrm{e}^{-2\varepsilon\sqrt{r}}$, we find

$$v \sim \mathrm{e}^{-2\min(\varepsilon,\lambda_T)\sqrt{r}}.$$

Conclusion: $v \sim \mathrm{e}^{-2\lambda_T\sqrt{r}}$ where $\lambda_T = \sqrt{\frac{2}{\mathrm{Im}\ \tau}}$

# Example: The four-punctured sphere

**Theorem [F-Mazzeo-Swoboda-Weiss]**

Let $\mathcal{M}$ be a (strongly-parabolic) $SU(2)$ Hitchin moduli space for the four-punctured sphere. The rate of exponential decay for the Hitchin moduli space is as Gaiotto-Moore-Neitzke conjecture:

$$g_{L^2} - g_{\mathrm{sf}} = O(\mathrm{e}^{-2\sqrt{\frac{2}{\mathrm{Im}\,\tau}}\sqrt{r}}),$$

$(\mathcal{M}, g_{L^2})$ is an ALG metric asymptotic to the model metric $g_{\mathrm{sf}}$.

Thank you!