

FURTHER SMALL CORRECTIONS AND EXPLANATIONS FOR “A CYLINDRICAL REFORMULATION OF HEEGAARD FLOER HOMOLOGY”

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This document contains further mild corrections and explanations for [Lip06], beyond those corrected in [Lip14].

Page 26 of [Lip14]. In the proof of Proposition 4.2', the instances to $\pi_{\mathbb{D}} \circ u$ should be $\pi_{\Sigma} \circ u$.

(Thanks to Morgan Weiler for pointing out this typo.)

Page 997 of [Lip06]. The argument in the top half of the page rules out the analogue of disk bubbling in the symmetric product, but not the analogue of sphere bubbling at points in $\Sigma \times \{0, 1\} \times \mathbb{R}$. To avoid sphere bubbling, the easiest solution is to use the tautological correspondence and the argument in [OSz04, Lemma 3.13] (which involves considering the projection from the symmetric product to the Picard torus). That is, by Ozsváth-Szabó's argument, for a generic perturbation of the α -circles, there is no $Sym^g(j_{\Sigma})$ -holomorphic sphere through any point in $T_{\alpha} \cap T_{\beta}$. The tautological correspondence and a rescaling argument then imply that there is no corresponding degeneration for maps to $\Sigma \times [0, 1] \times \mathbb{R}$.

(Thanks to Ian Zemke and Kristen Hendricks for point out this mistake.)

Page 1001 of [Lip06] In Proposition 8.6, the proof that A_{ζ} induces an action of the exterior algebra is incorrect: the moduli spaces $\widehat{\mathcal{M}}_{K,2}^A$ have an unaccounted for end where $p_1 \rightarrow p_2$. To correct the proof, we first show that for any $\zeta, \eta \in H_1(Y)$, the map $A_{\zeta} \circ A_{\eta} + A_{\eta} \circ A_{\zeta} = 0$ on \widehat{HF} , HF^+ , HF^- , and HF^{∞} . To see this, choose disjoint knots $K_{\zeta}, K_{\eta} \subset \Sigma \times [0, 1]$ representing ζ and η , and consider the index 2 moduli space of holomorphic curves with one point mapped to ζ and a second point mapped to η . The ends of this moduli space show that $A_{\zeta} \circ A_{\eta} + A_{\eta} \circ A_{\zeta}$ is chain homotopic to the zero map. Next, to see that $A_{\zeta}^2 = 0$ on Floer homology it suffices to consider the case that ζ is represented by a chain K in Σ which is dual to some α_i , i.e., K intersects α_i in one point and is disjoint from α_j for $j \neq i$. Let K' be a small isotopic translate of K , and consider the moduli space of holomorphic curves

$$\{u: (S, p, q) \rightarrow \Sigma \times [0, 1] \times \mathbb{R} \mid \pi_{\Sigma}(u(p)) \in K, \pi_{\Sigma}(u(q)) \in K', \pi_{\mathbb{R}}(u(p)) - \pi_{\mathbb{R}}(u(q)) > 0\}$$

(and with u satisfying the conditions (M0)–(M6) from the paper). This moduli space has no end with $\pi_{\mathbb{R}}(u(p)) - \pi_{\mathbb{R}}(u(q)) \rightarrow 0$ because K and K' are disjoint and intersect the α -circles in a single point. Then, it is easy to see that the ends of the moduli space imply that A_{ζ}^2 is chain homotopic to 0.

(Thanks to Ian Zemke for pointing out this mistake.)

Page 1005 of [Lip06]. In the proof of Lemma 9.3, the fact that the ends of $\widehat{\mathcal{M}}_1(\vec{x}^1, \vec{y}^2, k)$ correspond to height 2 holomorphic buildings in which the \mathbb{R} -invariant level has $\text{ind} = 1$ and

the non- \mathbb{R} -invariant level has $\text{ind} = 0$ is not sufficiently justified, because Proposition 4.2 was only proved with respect to \mathbb{R} -invariant almost complex structures. The easiest solution is to define Φ to only count embedded, rigid holomorphic curves in homology classes with $\text{ind} = 0$. (This is, in some sense, three conditions: the combinatorial index $\text{ind}(A) = e(A) + n_{\bar{x}}(A) + n_{\bar{y}}(A) = 0$, the curve must be embedded, and the curve must lie in a 0-dimensional moduli space. Presumably the condition that $\text{ind}(A) = 0$ implies the other two, but that has not been shown for non- \mathbb{R} -invariant almost complex structures.) Similarly, define $\widehat{\mathcal{M}}_1(\bar{x}^1, \bar{y}^2, k)$ to consist of $\text{ind}(A) = 1$, 1-dimensional moduli spaces of embedded curves with $n_3 = k$. Since $\text{ind}(A)$ agrees with the dimension of the moduli space of curves for \mathbb{R} -invariant levels and is additive under gluing, if a sequence of curves in $\widehat{\mathcal{M}}_1(\bar{x}^1, \bar{y}^2, k)$ converges to a 2-story holomorphic building then the \mathbb{R} -invariant level must have $\text{ind}(A) = 1$, so the non- \mathbb{R} -invariant level must have $\text{ind}(A) = 0$. Note also that gluing preserves embeddedness and non-embeddedness. It follows that the ends of $\bigcup_k \widehat{\mathcal{M}}_1(\bar{x}^1, \bar{y}^2, k)$ correspond to the terms in $\partial \circ \Phi + \Phi \circ \partial$, as desired.

(Thanks to Cagatay Kutluhan for pointing out this gap.)

Page 1022 of [Lip06]. In the formula defining K , the last instance of $B_{\alpha, \gamma}$ should be $B_{\alpha, \beta}$.

(Thanks to Michael Gartner for pointing out this mistake.)

Page 1043 of [Lip06]. The explicit description of the Hamiltonian H is incorrect. Figure 14 is correct, and clearer. Finding a correct formula for such a Hamiltonian is an easy exercise (which the author solved incorrectly). An explanation of why the resulting cylinders are Lagrangian with respect to a suitable symplectic form is also missing; see, for instance, [LOT14, Formula (3.25)]. (This is also relevant to the proof of isotopy invariance in Section 9.)

The labels w_i^+ and w_i^- in Figure 15 also seem to be reversed (if the author still understands the notation), making the proof of Lemma 11.8 unconvincing. A more convincing proof is the fact that the 1-gon maps are homogeneous with respect to the Maslov grading and, by any model computation (e.g., in genus 1) the top-graded generator is an output of the 1-gon maps. See the proof of Proposition 11.4.

(Thanks to Thomas Hockenhull for pointing out these mistakes.)

Page 1051 of [Lip06]. In the enumerated list at the bottom of the page, point (2) says that the S'_i are closed surfaces, which contradicts point (5). The S'_i should be the union of a closed surface with a bigon.

(Thanks to Michael Gartner for pointing out this mistake.)

Page 1057 of [Lip06].

In the proof of Lemma 13.3, the second and third paragraphs are wrong: pushing forward a pseudoholomorphic map via the t -dependent identification of U_i with the unit disk does not give a pseudoholomorphic map with respect to the split complex structure, because the derivative of the diffeomorphism contributes to the derivative of the pushforward map (i.e., because of the chain rule).

One solution is to work only with almost complex structures which are sufficiently close to split complex structures. For split complex structures, Lemma 13.3 follows from elementary complex analysis, and the condition of Lemma 13.3 is closed. So, by Gromov compactness, Lemma 13.3 holds for all almost complex structures in some neighborhood of the split ones.

One then restricts to this neighborhood of the split complex structures in the rest of the section.

For the rest of the proof of Proposition 13.2, apply the tautological correspondence in the complement of the diagonal, and then removal of singularities in a neighborhood of the diagonal.

The first paragraph of the proof of Proposition 13.7 on page 1059 contains the same mistake.

Another approach to this kind of comparison between the cylindrical and symmetric product or Hilbert scheme formulation can be found in Mok-Smith’s paper [MS19].

(Thanks to Paolo Ghiggini for pointing out this mistake.)

REFERENCES

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