Khovanov Homology Detects Split Links

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Outline

• History, statement of main theorem
• Overview of proof.
• Background on $A_\infty$-modules.
• The basepoint action on Khovanov homology.
• Twisted coefficients and action of the exterior algebra on Heegaard Floer homology
Khovanov homology detects...
Khovanov homology detects...

- [Grigsby-Wehrli, ’08] $\dim Kh(n - cable of K)$ detects the unknot $\forall n > 1$.
- [Hedden, ’08] $\dim Kh(2 - cable of K)$ detects the unknot.
- [Kronheimer-Mrowka, ’10] $\dim Kh(K)$ detects the unknot.
- [Hedden-Ni, ’10] $g \dim Kh(L)$ detects the 2-component unlink.
- [Hedden-Ni, ’12] $Kh(L)$ as a module over $\mathbb{F}_2[X_1, ..., X_\ell]/(X_1^2, ..., X_\ell^2)$ detects the unlink.
- [Batson-Seed, ’13] $g \dim Kh(L)$ detects the unlink.
- [Baldwin-Sivek, ’18] $Kh(L; \mathbb{Z})$ detects the trefoils.
- [Baldwin-Sivek-Xie, ’18] $g \dim Kh(L; \mathbb{F}_2)$ detects the Hopf link.
- [Xie-B. Zhang, ’19] $\dim Kh(L; \mathbb{F}_2)$ detects forests of Hopf links and unknots.
- [J. Wang, ’20] $\dim Kh(L; \mathbb{F}_2)$ detects trivial band sums $K_1 \bigoplus K_2$ of split links, among all band sums of $K_1 \bigoplus K_2$.

Near-universal strategy: exploit a spectral sequence to a more geometric invariant (Heegaard / monopole Floer homology or instanton Floer homology)
Khovanov homology [also] detects...split links.

• Given a link $L$, $p, q \in L$, can
  • form reduced Khovanov homology $\widetilde{Kh}(L)$ using the basepoint $p$
  • which is a module over $R = \mathbb{F}_2[X]/(X^2)$ using the basepoint $q$.

• **Theorem.** [L–Sarkar, ’19] Given a link $L$ and points $p, q \in L$, there is a 2-sphere in $S^3 \setminus L$ separating $p$ and $q$ if and only if $\widetilde{Kh}(L)$ is a free $R = \mathbb{F}_2[X]/(X^2)$-module.

• (For the rest of the talk, everything is with $\mathbb{F}_2$-coefficients.)
Equivalent conditions to the main theorem

• Call a chain complex $C_*$ over $R$ quasi-free if $C_*$ is quasi-isomorphic to a bounded chain complex of free $R$-modules.

• An obstruction to quasi-freeness over $R = \mathbb{F}_2[X]/(X^2)$: is total complex of
  \[
  \cdots \leftarrow C_* \leftarrow C_* \leftarrow C_* \leftarrow \cdots
  \]
acyclic? Call the homology of this the \textit{unrolled homology}.

• \textbf{Theorem.} [L-Sarkar] For $p, q \in L$, the following are equivalent:
  1. There is a 2-sphere separating $p$ and $q$.
  2. $\overline{Kh}(L)$ is a free $R$-module.
  3. $\tilde{C}^{Kh}(L)$ is quasi-free.
  4. The unrolled homology of $\tilde{C}^{Kh}(L)$ is trivial.
Overview of the proof
Ingredients

- **Freeness.** [Shumakovitch ’03] $Kh(K)$ (unreduced) is a free $R = \mathbb{F}_2[X]/(X^2)$-module.
  
  - Also follows from an argument in odd Khovanov homology (Ozsváth-Rasmussen-Szabó).

- **Basepoint Independence.** Up to quasi-isomorphism over $R$, $\tilde{C}^{Kh}(L)$ depends only on $L$ and which components contain $p, q$.

- **Spectral Sequence.** The Ozsváth-Szabó spectral sequence $\tilde{Kh}(L) \Rightarrow \widehat{HF}(\Sigma(L))$ respects the $(A_\infty)$ $R$-module structure.

- **Unrolled $\widehat{HF}$.** The unrolled homology of $\widehat{HF}(Y)$ is isomorphic to the Novikov twisted Heegaard Floer homology $\widehat{HF}(Y; \Lambda_\omega)$.

- **Sphere Detection.** $\widehat{HF}(Y; \Lambda_\omega)$ detects homologically essential $S^2$s. Specifically, $\widehat{HF}(Y; \Lambda_\omega) = 0$ if and only if there is an $S^2$ with $\langle \omega, S^2 \rangle \neq 0$.

**TFAE:**

1. There is a 2-sphere separating $p$ and $q$.
2. $\tilde{Kh}(L)$ is a free $R$-module.
3. $\tilde{C}^{Kh}(L)$ is quasi-free.
4. The unrolled homology of $\tilde{C}^{Kh}(L)$ is trivial.
Steps

• 1 ⇒ 2, 1 ⇒ 3: For a split diagram, immediate from freeness of \( Kh(K) \) and Künneth theorem. For a general diagram, follows from basepoint independence.

• 2 ⇒ 4, 3 ⇒ 4: Algebra:
  • Can compute unrolled homology from \( A_\infty \)-module structure on \( \tilde{Kh}(L) \).
  • Consider the filtration on \( \cdots \leftarrow C_* \leftarrow C_* \leftarrow C_* \leftarrow \cdots \) from grading on \( C_* \).

• 4 ⇒ 1:
  • Spectral sequence: Unrolled homology of \( \tilde{C}^{Kh}(L) \) trivial implies unrolled homology of \( \widetilde{HF}(\Sigma(L)) \) trivial.
  • Unrolled homology: this is equivalent to \( \widetilde{HF}(\Sigma(L); \Lambda_\omega) = 0 \).
  • Sphere detection: this is equivalent to existence of a splitting sphere.

TFAE:
1. There is a 2-sphere separating \( p \) and \( q \).
2. \( \tilde{Kh}(L) \) is a free \( R \)-module.
3. \( \tilde{C}^{Kh}(L) \) is quasi-free.
4. The unrolled homology of \( \tilde{C}^{Kh}(L) \) is trivial.
$A_\infty$ background
\[ A_\infty \]-module basics

- An \( A_\infty \)-module over \( R \) is a (graded) vector space \( M \) and maps \( m_{1+n} : M \otimes R^\otimes n \to M \) satisfying some compatibility conditions.
- For \( R = \mathbb{F}_2[X]/(X^2) \) (and \( M \) strictly unital) these are maps
  \[
  m_{1+n}(\cdot, X, \cdots, X) : M \to M
  \]
  \[
  \sum_{i+j=n} m_{1+i} \circ m_{1+j} = 0
  \]
- The operation \( m_1 \) is a differential on \( M \).
- A chain complex of \( R \)-modules gives an \( A_\infty \)-module with \( m_{1+n} = 0 \) for \( n > 1 \).
$A_\infty$-module basics

- **Homological Perturbation Lemma.** Given an $A_\infty$-module $M$, a chain complex $N$, and a chain homotopy equivalence $M \simeq N$ over $\mathbb{F}_2$ there is an induced $A_\infty$-module structure on $N$ so that $M$ and $N$ are homotopy equivalent $A_\infty$-modules.

- In particular, homology of any chain complex of $R$-modules is an $A_\infty$-module.

- **Derived category is the $A_\infty$ homotopy category.** Given chain complexes of $R$-modules $M$ and $N$, $M$ and $N$ are quasi-isomorphic chain complexes of $R$-modules if and only if they are homotopy equivalent as $A_\infty$-modules.
Unrolled complex of an $A_\infty$-module

• The unrolled complex of $M$ is $M \otimes_{\mathbb{F}_2} \mathbb{F}_2[Y^{-1}, Y]$ with differential
  \[
  \partial (x \otimes Y^i) = \sum_{n} m_{1+n}(x, X, \ldots, X)Y^{i+n}.
  \]

• Clearly invariant under $A_\infty$ homotopy equivalence. So, by homological perturbation lemma, unrolled complex of $\widetilde{K}h(L)$ and $\widetilde{C}^{Kh}(L)$ agree.

• $2 \Rightarrow 4$ obvious. $3 \Rightarrow 4$ follows by filtering by grading on $M$.

1. There is a 2-sphere separating $p$ and $q$.
2. $\widetilde{K}h(L)$ is a free $R$-module.
3. $\widetilde{C}^{Kh}(L)$ is quasi-free.
4. The unrolled homology of $\widetilde{C}^{Kh}(L)$ is trivial.
The basepoint action on Khovanov homology
Invariance of the module structure.

• Fix points \( p, q \in L \). Endows \( C^{Kh}(L) \) with structure of a \((\mathbb{F}_2[W]/(W^2), \mathbb{F}_2[X]/(X^2))\)-bimodule (or \( \mathbb{F}_2[W, X]/(W^2, X^2) \)-module).

• **Theorem.** [Hedden-Ni; LS] Up to quasi-isomorphism, the differential bimodule \( C^{Kh}(L) \) depends only on the components containing \( p, q \).

• **Corollary.** Up to quasi-isomorphism, the module structure on \( \hat{C}^{Kh}(L) \) and \( A_{\infty} \)-module structure on \( Kh(L) \) and \( \hat{Kh}(L) \) depend only on the components containing \( p, q \).

• To prove the theorem, it suffices to construct an \( A_{\infty} \) homotopy equivalence (or quasi-isomorphism) associated to moving a basepoint through a crossing.
Solid: differential. Dashed: $m_2(W, \cdot)$. Dotted: $m_2(\cdot, X)$. Double: $f_{0,1,0}$. Double-dashed: $f_{0,1,1}(\cdot, X)$.
Module structures on Heegaard Floer homology
A tale of two twistings

• Fix $Y^3$, homomorphism $\omega: H_2(Y) \to \mathbb{Z}$.

• Ozsváth–Szabó construct:
  • An action of $\Lambda^* \mathbb{F}_2 = \mathbb{F}_2[X]/(X^2)$ on (untwisted) $\widehat{HF}(Y)$. (More generally, a $\Lambda^*(H_1(Y)/\text{tors})$-action.)
  • Twisted $\widehat{HF}(Y; \Lambda_\omega)$, a module over $\mathbb{F}_2[t^{-1}, t]$ or $\mathbb{F}_2[t^{-1}, t]$

• Ni, Hedden-Ni, Alishahi-L: $\widehat{HF}(Y; \Lambda_\omega)$ vanishes if and only if $Y$ has an $S^2$ with $\omega([S^2]) \neq 0$.

• Hedden-Ni, LS: The spectral sequence $\widehat{Kh}(L) \Rightarrow \widehat{HF}(\Sigma(L))$ respects the $\mathbb{F}_2[X]/(X^2)$-action.

• Goal: relate $\widehat{HF}(Y)_{\mathbb{F}_2[X]/(X^2)}$ and $\widehat{HF}(Y; \Lambda_\omega)_{\mathbb{F}_2[t^{-1}, t]}$.

• (cf. earlier work of Sarkar, work of Zemke.)
The $H_1/torsion$-action

• Differential on $\widehat{HF}(\Sigma, \alpha, \beta, z)$:
  \[
  \partial x = \sum_y \sum_{\phi \in \pi_2(x,y)} \left( #M(\phi) \right) y \\
  \frac{\partial x}{\partial \phi} = \sum_y \sum_{\phi \in \pi_2(x,y)} \left( #M(\phi) \right) \zeta \partial \alpha \phi \frac{y}{y} \\
  \frac{\partial y}{\partial \phi} = 0 \\
  n_z(\phi) = 0 \\
  \mu(\phi) = 1
  \]

• Action of $\zeta \in H_1(Y)$:
  \[
  x \cdot \zeta = \sum_y \sum_{\phi \in \pi_2(x,y)} \left( #M(\phi) \right) (\zeta \cdot \partial \alpha \phi) y \\
  \frac{x \cdot \zeta}{x} = y \\
  y \cdot \zeta = 0
  \]
The $A_\infty H_1/torsion$-action

- Action of $\zeta \in H_1(Y)$:

$$x \cdot \zeta = \sum_y \sum_{\substack{\phi \in \pi_2(x,y) \\ n_2(\phi) = 0 \\ \mu(\phi) = 1}} (#M(\phi))(\zeta \cdot \partial_\alpha \phi) y$$

- Equivalently, $x \cdot \zeta$ counts disks $u: [0,1] \times \mathbb{R} \to Sym^g(\Sigma)$ with $u(1,0) \in \zeta \times Sym^{g-1}(\Sigma)$.

- At the level of homology, $(x \cdot \zeta) \cdot \zeta = 0$ by considering 1D moduli space of $u$ with $u(1,0) \in \zeta$, $u(1,t) \in \zeta'$ for some $t > 0$ ($\zeta'$ a pushoff of $\zeta$).

$\partial(x) = 2y = 0$
$\partial(y) = 0$

$x \cdot \zeta = y$
$y \cdot \zeta = 0$
The $A_\infty H_1/torsion$-action

- Action of $\zeta \in H_1(Y)$:

$$x \cdot \zeta = \sum_y \sum_{\phi \in \pi_2(x,y) \atop n_2(\phi)=0, \mu(\phi)=1} (#M(\phi))( \zeta \cdot \partial_a \phi ) y$$

- Equivalently, $x \cdot \zeta$ counts disks $u: [0,1] \times \mathbb{R} \to \text{Sym}^g (\Sigma \setminus \{z\})$ with $u(1,0) \in \zeta \times \text{Sym}^{g-1}(\Sigma)$.

- At the level of homology, $(x \cdot \zeta) \cdot \zeta = 0$ by considering 1D moduli space of $u$ with $u(1,0) \in \zeta$, $u(1,t) \in \zeta'$ for some $t > 0$ ($\zeta'$ a pushoff of $\zeta$).

- Define $m_3(x, \zeta, \zeta)$ by counting 0D moduli space of this form.

- Define $m_n(x, \zeta, ..., \zeta)$, $n > 3$, similarly.

partial equations:

$$\partial(a) = 2b + 2c = 0$$
$$\partial(b) = d$$
$$\partial(c) = d$$
$$\partial(d) = 0$$

$$a \cdot \zeta = b + 2b + c + 2c = b + c$$
$$b \cdot \zeta = d$$
$$c \cdot \zeta = 0$$
$$d \cdot \zeta = 0$$

$$m_3(a, \zeta, \zeta) = b + 2b + 2c = b$$
$$m_3(\partial(a), \zeta, \zeta) + \partial(m_3(a, \zeta, \zeta)) = 0 + d$$
Twisted coefficient $\hat{HF}$

- Fix $Y^3$, homomorphism $\omega: H_2(Y) \to \mathbb{Z}$.
- Choose $\zeta \in H_1(\Sigma)$ with $\omega(B) = \zeta \cdot \partial_\alpha(B)$.
- Define
  \[
  \partial x = \sum_y \sum_{\phi \in \pi_2(x,y)} (\#M(\phi)) t^{(-\partial_\alpha \phi)} y.
  \]
- (This is a module over $\mathbb{F}_2[t^{-1}, t]$.)
- Notice that
  \[
  \partial = \partial \bigg|_{t=1}.
  \]

\[
\begin{align*}
\partial(x) &= t^1 y + t^0 y \\
\partial(y) &= 0
\end{align*}
\]
First relation: Hasse derivatives

- $\partial x = \sum_y \sum_{\phi \in \pi_2(x, y)} (\#M(\phi)) t^{(\zeta \cdot \partial_\alpha \phi)} y$.
  
- $x \cdot \zeta = \sum_y \sum_{\phi \in \pi_2(x, y)} (\#M(\phi)) (\zeta \cdot \partial_\alpha \phi) y = \frac{d}{dt} |_{t=1} (\partial x)$

- $\frac{d^2}{dt^2} = 0$ over $\mathbb{F}_2$. But there’s an analogue $D^2$ of $\frac{1}{2} \left( \frac{d^2}{dt^2} \right)$, called the Hasse derivative. (More generally, $D^n$ is an analogue of $\frac{1}{n!} \left( \frac{d^n}{dt^n} \right)$.)

- **Proposition.** \([\text{LS}]\) $m_{1+n}(x, \zeta, \cdots, \zeta) = D^n |_{t=1} (\partial x)$.

- **Corollary.** \([\text{LS}]\) The unrolled homology of $\tilde{CF}(Y)$ (or $\tilde{HF}(Y)$) is isomorphic to $\tilde{HF}(Y; \Lambda_\omega)_{\mathbb{F}_2[t^{-1}, t]}$ (so detects homologically essential $S^2$'s).
Second relation: Koszul duality

- Can view $\mathbb{F}_2$ as an $\mathbb{F}_2[t^{-1}, t]$-module where $t$ acts as 1.
- There is an isomorphism of algebras (or $A_\infty$-algebras)
  $$\mathbb{F}_2 [X] / (X^2) \cong \text{Ext}_{\mathbb{F}_2[t^{-1}, t]}(\mathbb{F}_2, \mathbb{F}_2)$$
- **Proposition.** [LS] There is an isomorphism of $A_\infty$-modules over $\mathbb{F}_2 [X] / (X^2)$
  $$\widehat{HF}(Y) \cong \text{Tor}_{\mathbb{F}_2[t^{-1}, t]}(\widehat{CF}(Y), \mathbb{F}_2)$$
- **Proof.** See that tensoring with a free resolution of $\mathbb{F}_2$ leads to the formula with Hasse derivatives from the previous slide.
The Ozsváth-Szabó spectral sequence respects the $A_\infty$-module structure

**Proposition.** [LS] There is a filtered $A_\infty$-module $C$ so that:

1. As an unfiltered $A_\infty$-module, $C$ is quasi-isomorphic to $\widehat{CF}(\Sigma(L))$.
2. The differential strictly increases the filtration.
3. There is an isomorphism of modules $C \cong \hat{C}_{Kh}(L)$ taking filtration to homological grading.
4. To first order, the differential on $C$ agrees with the Khovanov differential.
5. To zeroth order, $m_2$ agrees with the action of $X$ on $\hat{C}_{Kh}(L)$.

**Corollary.** [LS] If the unrolled homology of the Khovanov complex is trivial then the unrolled homology of the $\widehat{CF}(\Sigma(L))$ is trivial.

In that case, $\Sigma(L)$ has a homologically essential $S^2$ so $L$ is split.
Review of the proof

• 1 ⇒ 2, 1 ⇒ 3: For a split diagram, immediate from Freeness of $Kh(K)$ and Künneth theorem. For a general diagram, follows from **Basepoint independence**.

• 2 ⇒ 4, 3 ⇒ 4: Algebra:
  • Can compute unrolled homology from $A_\infty$-module structure on $\widetilde{Kh}(L)$.
  • Consider the filtration on $\cdots \leftarrow C_* \leftarrow C_* \leftarrow C_* \leftarrow \cdots$ from grading on $C_*$.  

• 4 ⇒ 1:
  • **Spectral sequence**: Unrolled homology of $\tilde{C}^{Kh}(L)$ trivial implies unrolled homology of $\widehat{HF}(\Sigma(L))$ trivial.
  • **Unrolled homology**: this is equivalent to $\widehat{HF}(\Sigma(L); \Lambda_\omega) = 0$.
  • **Sphere detection**: this is equivalent to existence of a splitting sphere.
That’s all. Thanks for listening!