COMPUTING INVOLUTIVE $\widehat{HF}$

A bordered Floer homology story

Robert Lipshitz[1]
Joint with Kristen Hendricks[2]

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[2]Supported by NSF Grant DMS-1663778

Computing involutive $\widehat{HF}$: Outline

1. The answer. [Hendricks-L]
2. Review of bordered Floer homology. [L-Ozsváth-Thurston]
3. How to compute $\widehat{HF}(Y)$ using bordered Floer homology. [L-Ozsváth-Thurston]
4. Review of involutive Heegaard Floer homology. [Hendricks-Manolescu]
5. How to compute $\widehat{HFI}(Y)$ using bordered Floer homology. [Hendricks-L]
6. Bordered involutive Heegaard Floer homology. [Hendricks-L]
7. Application: surgery exact triangle for $\widehat{HFI}(Y)$. [Hendricks-L]

The answer

- $\widehat{HFI}(Y) = H_*(\widehat{CF}(Y) \xrightarrow{Id+\iota} \widehat{CF}(Y))$
- Choose a Heegaard splitting $Y = H_0 \cup H_1$.
- $\widehat{CF}(Y) = Mor\left(\widehat{CFD}(H_0), \widehat{CFD}(H_1)\right)$.
- Know how to compute $\widehat{CFD}(H_i)$ [L-Ozsváth-Thurston]
- Map ι is the composition

$$Mor\left(\widehat{CFD}(H_0), \widehat{CFD}(H_1)\right) \to Mor\left(\widehat{CFDA}(AZ) \otimes \widehat{CFD}(H_0), \widehat{CFDA}(AZ) \otimes \widehat{CFD}(H_1)\right)$$
$$\to Mor\left(\widehat{CFD}(H_0), \widehat{CFD}(H_1)\right)$$

  where first map is $f \mapsto Id \otimes f$ and second comes from homotopy equivalences $\widehat{CFDA}(AZ) \otimes \widehat{CFD}(H_i) \simeq \widehat{CFD}(H_i)$.
- In fact, this homotopy equivalence is unique, hence computable.

$\widehat{HF}(Y)$: what is it, who cares?

1. One of the variants of Heegaard Floer homology introduced by Ozsváth-Szabó (arXiv:math/0101206).
2. Defined via pseudoholomorphic curves in a high-dimensional symplectic manifold.
3. Detects minimal genus surfaces, whether 3-manifolds fiber over S^1. Information about tautness of foliations, tightness of contact structures, sliceness of knots. (Many papers, many authors.)
4. Applications to surgery questions, concordance, knot theory, contact topology.
5. **Theorem.** (Taubes, Kutluhan-Lee-Taubes, Colin-Ghiggini-Honda) $\widehat{HF}(Y)$ is the same as the non-S^1-equivariant Seiberg-Witten Floer homology of Y.
6. First algorithm for computing $\widehat{HF}(Y)$ due to Sarkar-Wang (arXiv:math/0607777).
   - *Remarkable, but slow –* $3^{b^{2^d}}$ *steps (Hales-Karabash-Lock,* arXiv:0711.4405*).*

Bordered Floer: structure

To an	Bordered Floer assigns a
Oriented surface decomposed into handles $F(\mathcal{Z})$	dg algebra $\mathcal{A}(\mathcal{Z})$ • Finite-dimensional over $\boldsymbol{F}_2$ • $\mathcal{A}(S^2) = \mathbf{F}_2$.

L-Ozsváth-Thurston,
arXiv:0810.0687
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Cobordism $Y\colon F(\mathcal{Z}_1) \to F(\mathcal{Z}_2)$	A_∞-bimodule $\widehat{CFDA}(Y)$ over $\left(\mathcal{A}(\mathcal{Z}_1), \mathcal{A}(\mathcal{Z}_2)\right)$.

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Cobordism $Y: F(\mathcal{Z}_1) \to F(\mathcal{Z}_2)$ • If $F(\mathcal{Z}_1) = S^2$ (resp. $F(\mathcal{Z}_2) = S^2$) • If $F(\mathcal{Z}_1) = F(\mathcal{Z}_2) = S^2$ • Y mapping cylinder of a diffeomorphism ϕ	A_∞-bimodule $\widehat{CFDA}(Y)$ over $\left(\mathcal{A}(\mathcal{Z}_1), \mathcal{A}(\mathcal{Z}_2)\right)$. • then denoted $\widehat{CFA}(Y)$ (resp. $\widehat{CFD}(Y)$) • then $\widehat{CFDA}(Y) = \widehat{CF}(Y \cup D^3 \cup D^3)$ • then $\widehat{CFDA}(Y)$ denoted $\widehat{CFDA}(\phi)$.

Theorem. (Pairing Theorem) If $Y_1: F(\mathcal{Z}_1) \to F(\mathcal{Z}_2)$ and $Y_2: F(\mathcal{Z}_2) \to F(\mathcal{Z}_3)$ then $\widehat{CFDA}\left(Y_1 \cup_{F(\mathcal{Z}_2)} Y_2\right) \simeq \widehat{CFDA}(Y_1) \otimes_{\mathcal{A}(\mathcal{Z}_2)} \widehat{CFDA}(Y_2)$.

- There are other variants of the bimodules, denoted $\widehat{CFDD}(Y)$, $\widehat{CFAA}(Y)$.
- $\widehat{CFDD}(Y)$ is easiest to compute.
- All variants carry equivalent information.

L-Ozsváth-Thurston, arXiv:0810.0687
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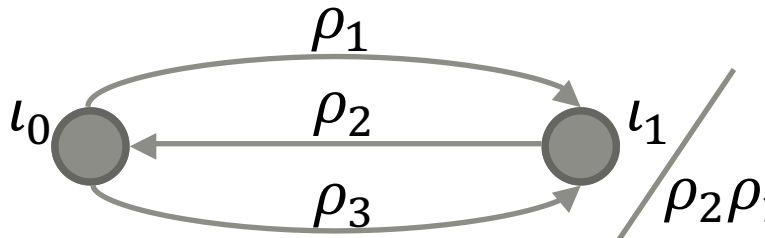
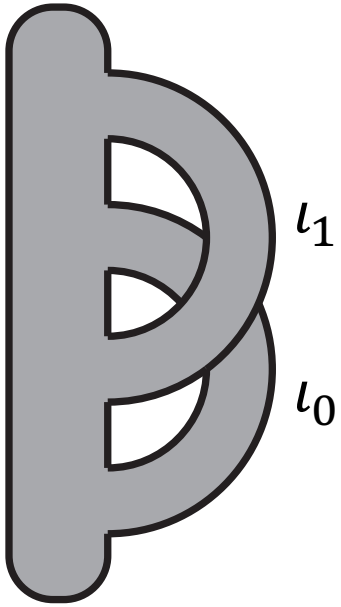
Some examples with torus boundary

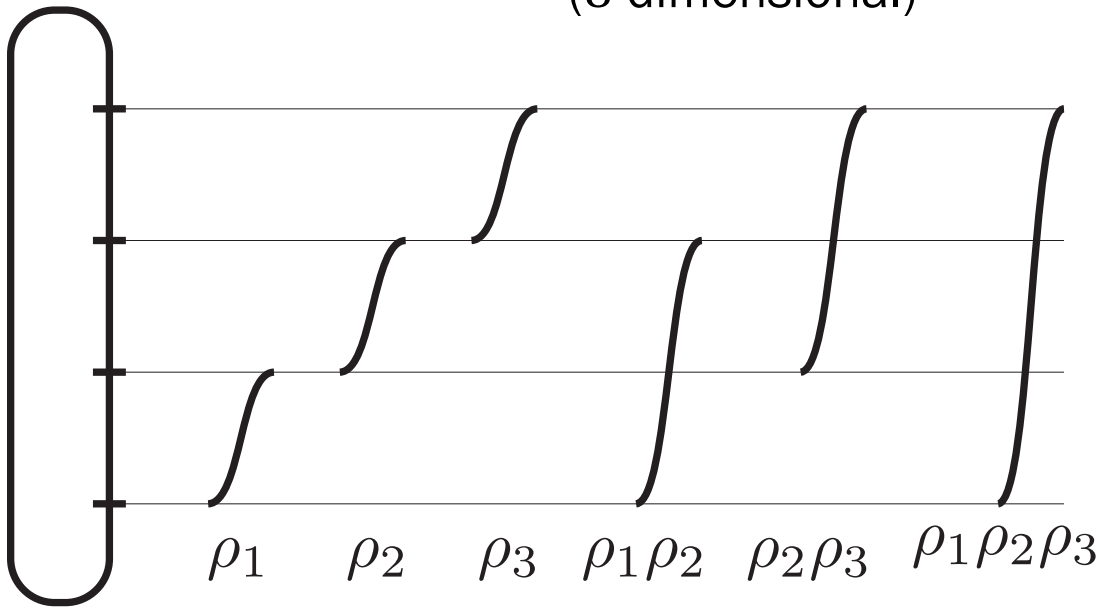
- The algebra associated to the torus:



$$\rho_2\rho_1 = \rho_3\rho_2 = 0$$

(8-dimensional)





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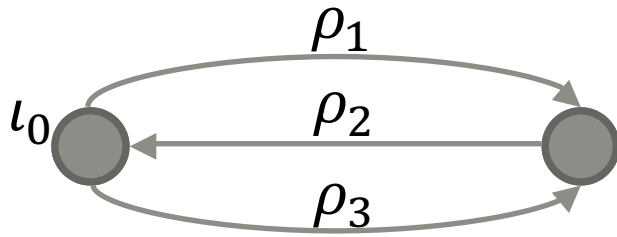

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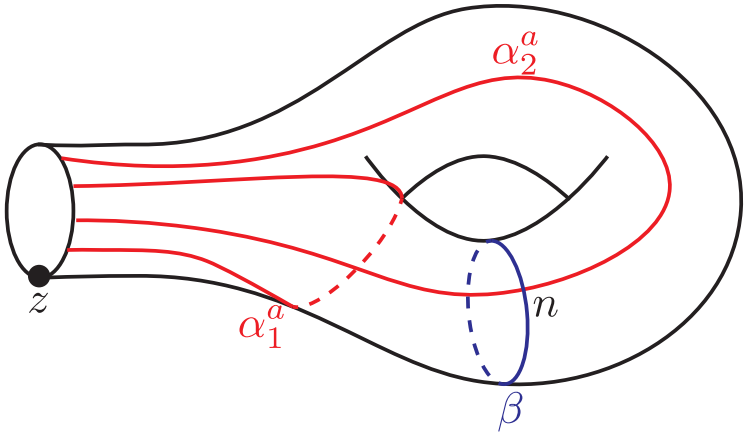
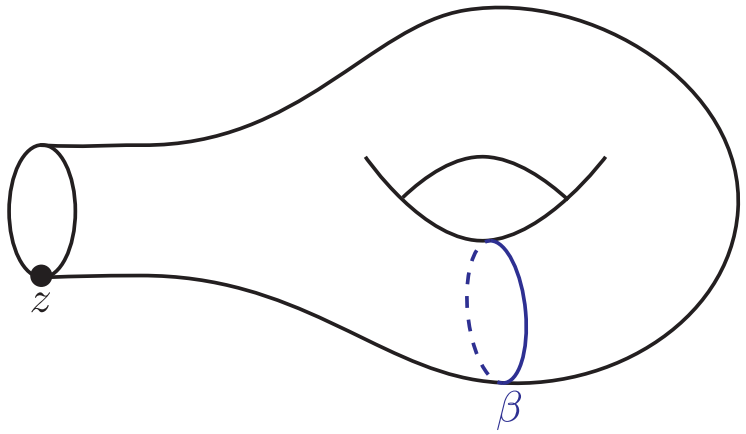
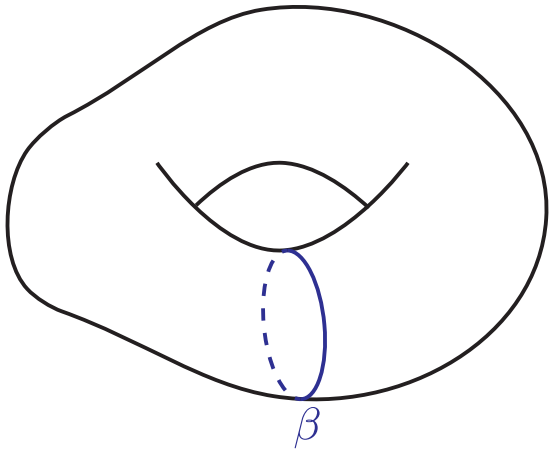
- Some modules associated to solid tori:



Bordered Heegaard diagram
for solid torus





Heegaard diagram
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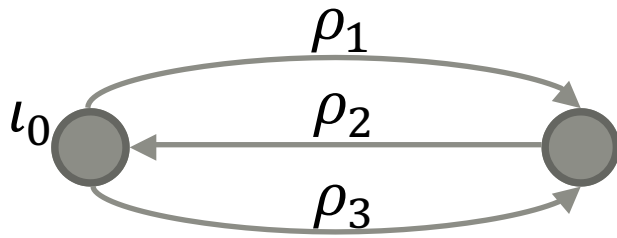

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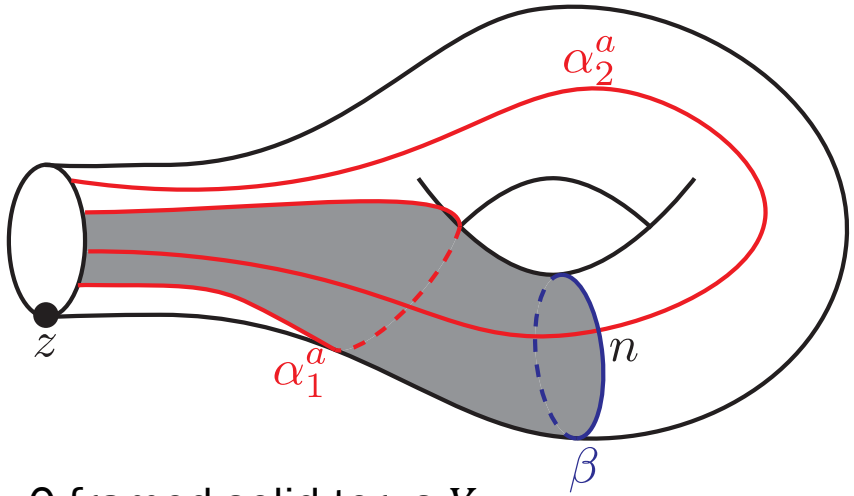
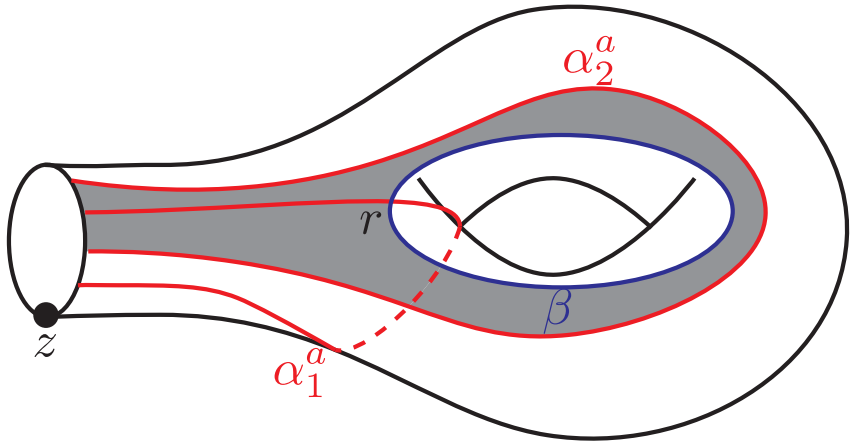


0-framed solid torus Y_0

$\widehat{CFD}(Y_0) = \mathcal{A}\iota_1\langle n\rangle,\ \partial(n) = \rho_1\rho_2 n.$



infinity-framed solid torus Y_∞

$\widehat{CFD}(Y_\infty) = \mathcal{A}\iota_0\langle r\rangle,\ \partial(r) = \rho_2\rho_3 \mathrm{r}.$

Some examples with torus boundary

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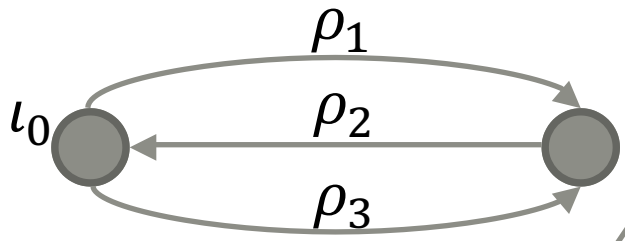

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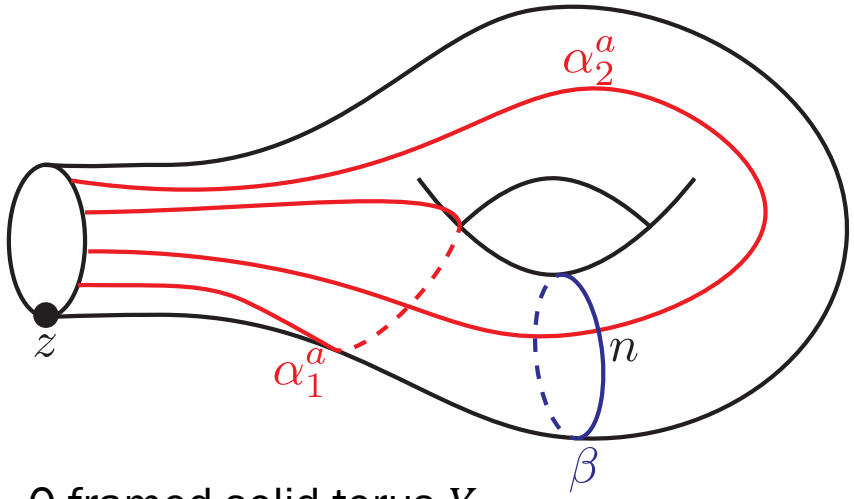

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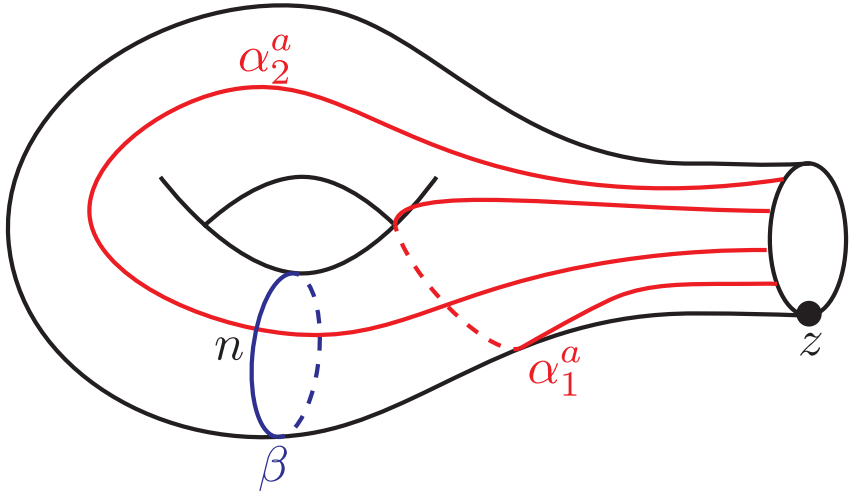


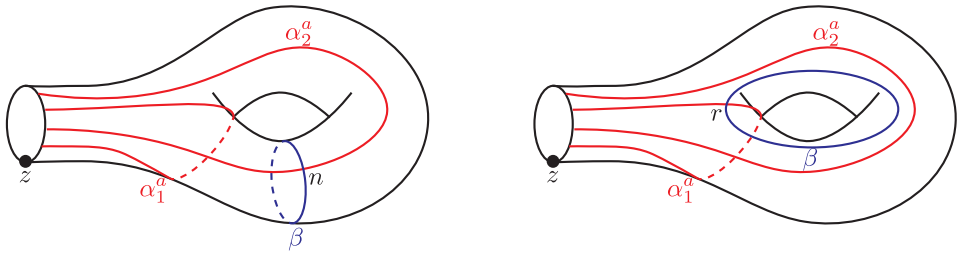
0-framed solid torus Y_0, "A" type

$\widehat{CFA}(Y_0) = \boldsymbol{F}_2\langle n\rangle$

$m_3(n, \rho_2, \rho_1) = n$
$m_4(n, \rho_2, \rho_1\rho_2, \rho_1) = n$
$m_5(n, \rho_2, \rho_1\rho_2, \rho_1\rho_2, \rho_1) = n$
…

Take-away points

- The algebras associated to surfaces are finite-dimensional and fairly concrete.
- There are lots of solid tori, because:
- A cobordism from $F(\mathcal{Z}_1)$ to $F(\mathcal{Z}_2)$ is:
  - A 3-manifold Y with boundary components $\partial Y = \partial_L Y \cup \partial_R Y$.
  - Diffeomorphisms $\phi_L: F(\mathcal{Z}_1) \to -\partial_L Y$ and $\phi_R: F(\mathcal{Z}_2) \to \partial_R Y$.
- So, a *bordered handlebody* is (handlebody H, diffeomorphism $\phi: F(\mathcal{Z}) \to \partial H$).
- The *mapping cylinder* of $\phi: F(\mathcal{Z}_1) \to F(\mathcal{Z}_2)$ is $([0,1] \times F(\mathcal{Z}_1), Id, \phi)$.
- Gluing on a mapping cylinder twists the boundary diffeomorphism.



Computing $\widehat{HF}$ by factoring

$\widehat{CFD}(H, \phi_0)$ for one particular bordered handlebody (H, ϕ_0) and
and $\widehat{CFDD}(\psi_i)$ for set of generators of the mapping class group(oid) (arcslides).

↓ Dualities

$\widehat{CFA}(H, \phi_0)$ and $\widehat{CFD}(H, \phi_0)$ for one particular bordered handlebody (H, ϕ_0)
and $\widehat{CFDA}(\psi_i)$ for set of generators of the mapping class group(oid) (arcslides).

Pairing theorem ↓

$\widehat{CFA}(H, \phi_0)$ and $\widehat{CFD}(H, \phi_0)$ for one particular bordered handlebody (H, ϕ_0)
and $\widehat{CFDA}(\psi)$ for (mapping cylinder of) any mapping class $\psi: F(\mathcal{Z}_1) \to F(\mathcal{Z}_2)$

↓ Pairing theorem

$\widehat{CFA}(H, \phi)$ and $\widehat{CFD}(H, \phi)$ for any bordered handlebody (H, ϕ)

Heegaard splitting, pairing theorem ↓

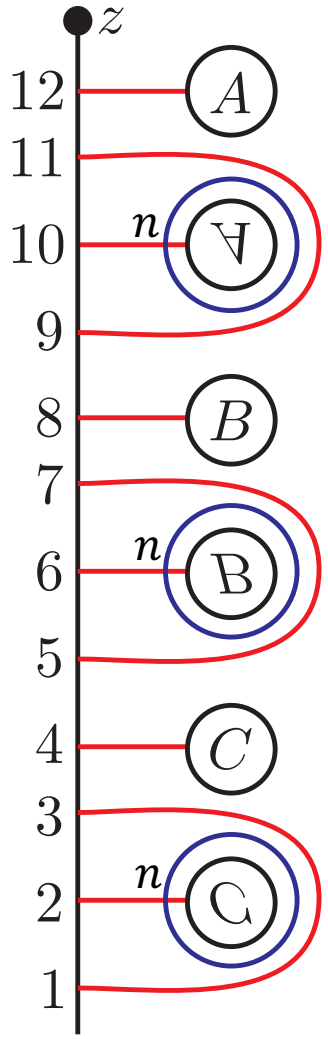
$\widehat{HF}(Y)$ for any 3-manifold Y

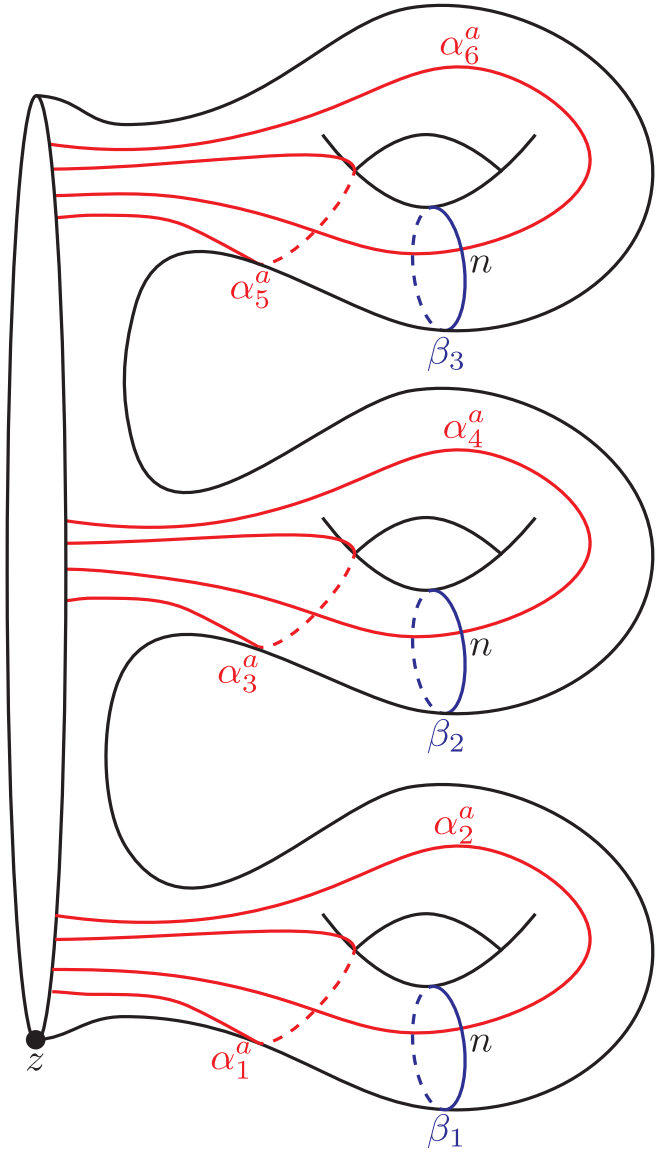
L-Ozsváth-Thurston, arXiv:1010.2550

Some details: modules for 0-framed handlebodies

$$\widehat{CFD}(Y_0) = \mathcal{A}(\mathcal{Z})\langle n \rangle$$

$$\partial(n) = (\rho_{1,3} + \rho_{5,7} + \rho_{9,11} + \cdots)n$$





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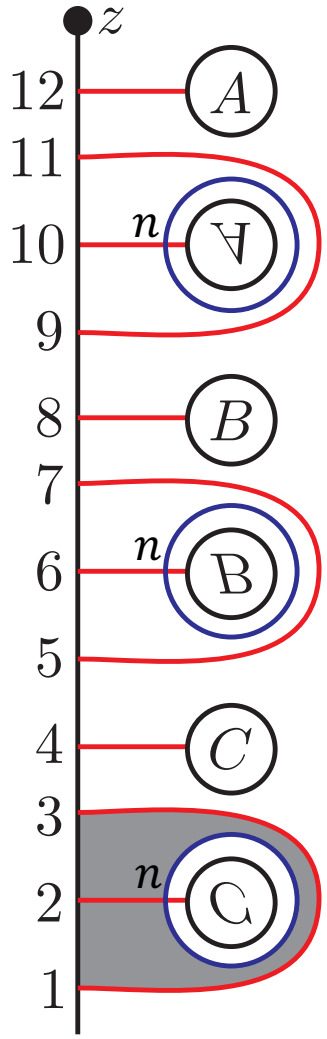

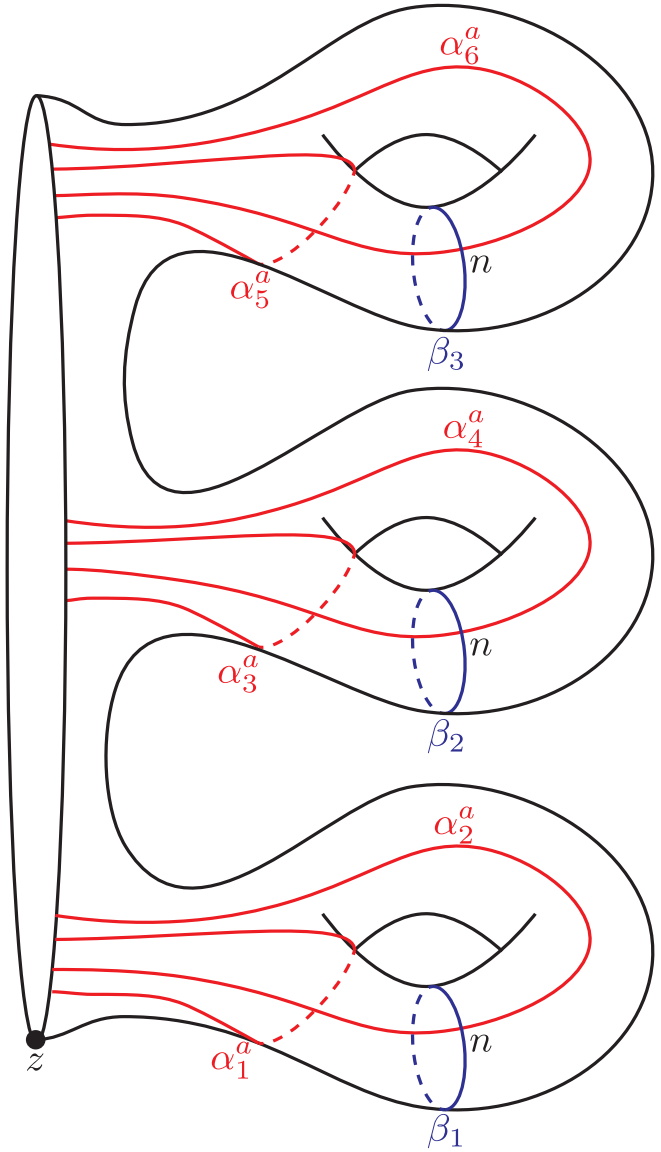

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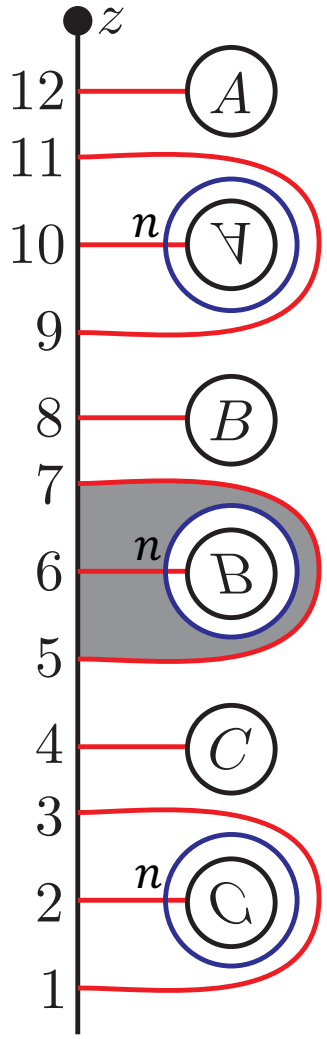
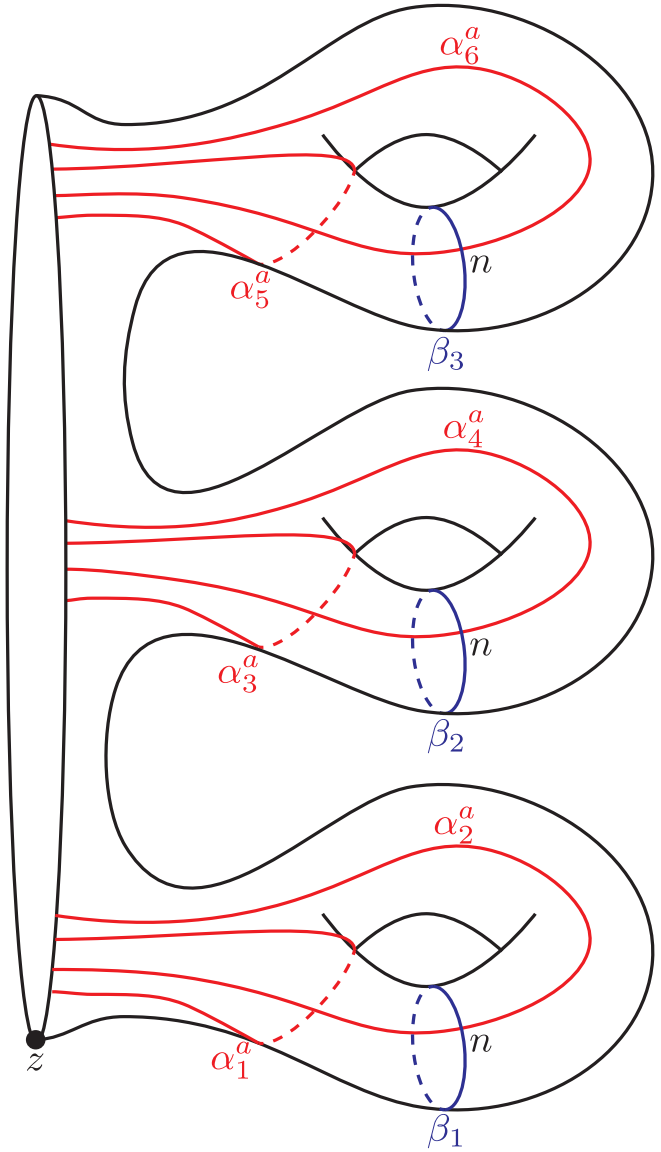
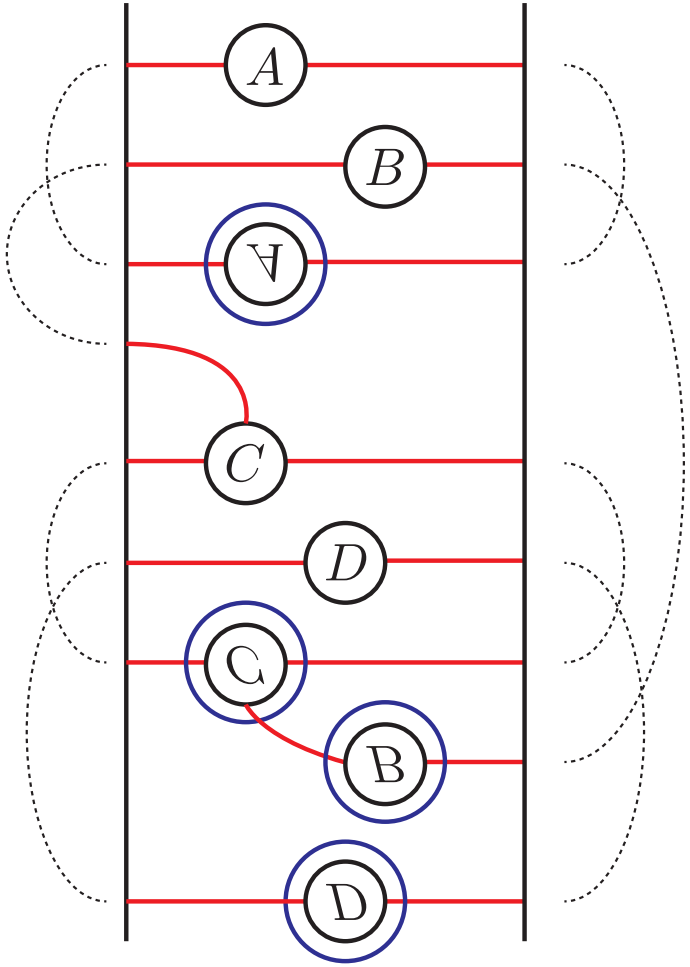
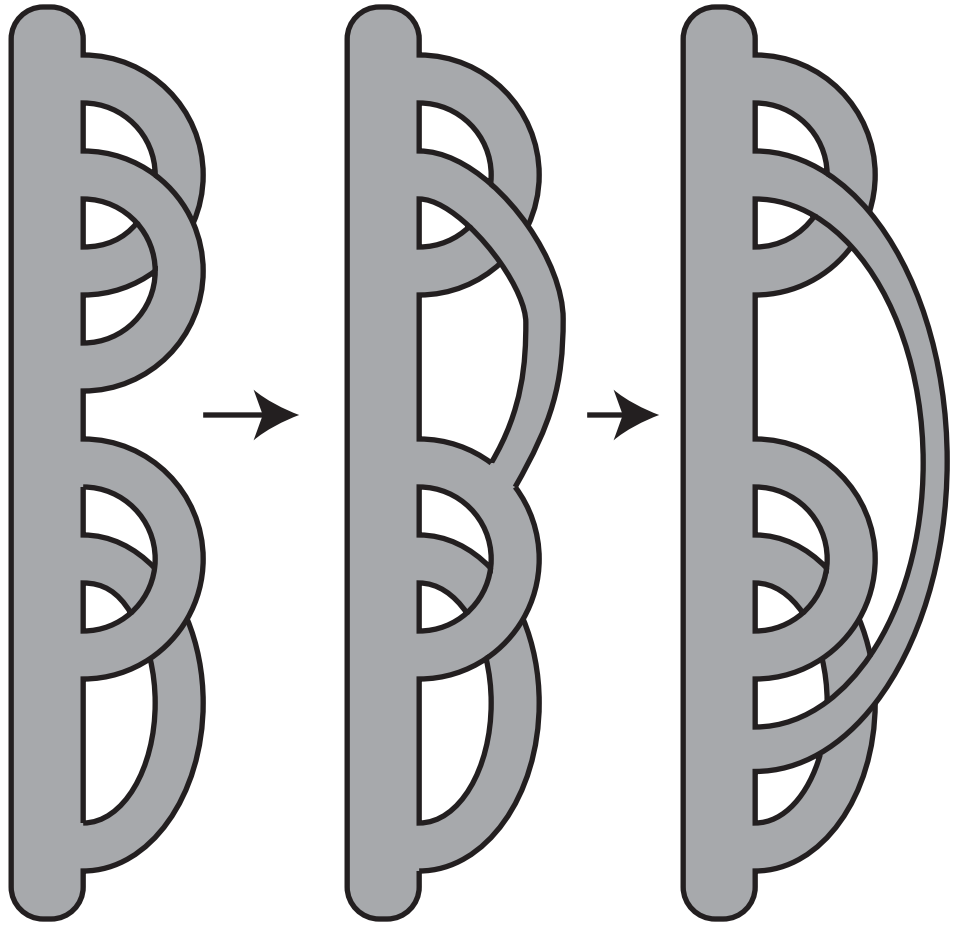


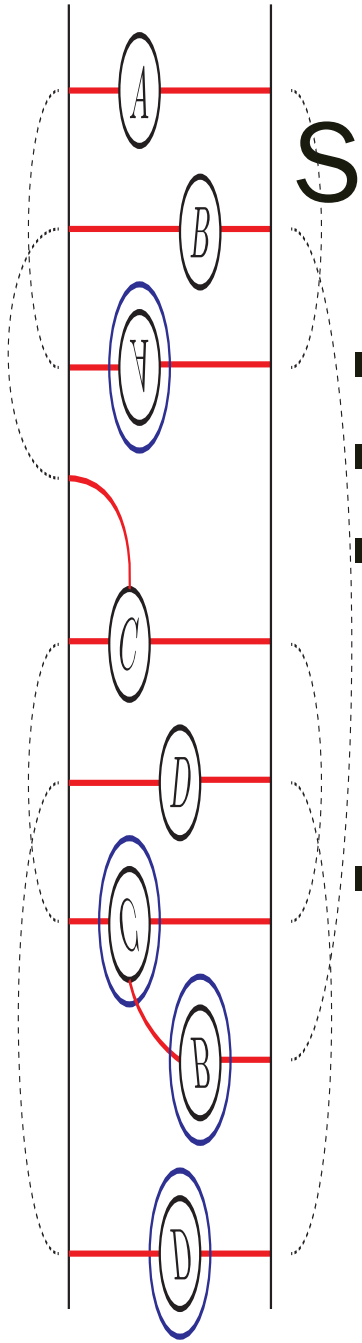


Some details: bimodules for arc slides

- The generators for the mapping class groupoid we work with are *arcslides*.







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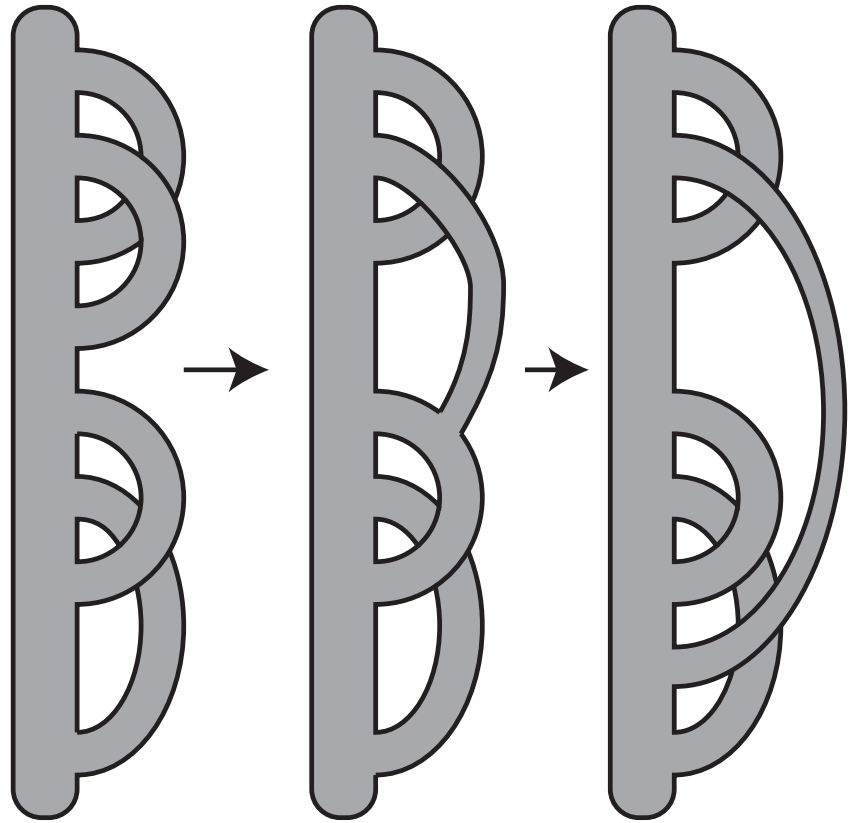
- (Relations among these generators were given by Bené in arXiv:0802.2747.)
- For ϕ and arcslide, $\widehat{CFDD}(\phi)$ is determined by:
  - *Its minimal set of generators.*
  - *It is invertible for tensor product.*
  - $\partial^2 = 0$.
- The answer is complicated, but combinatorial.



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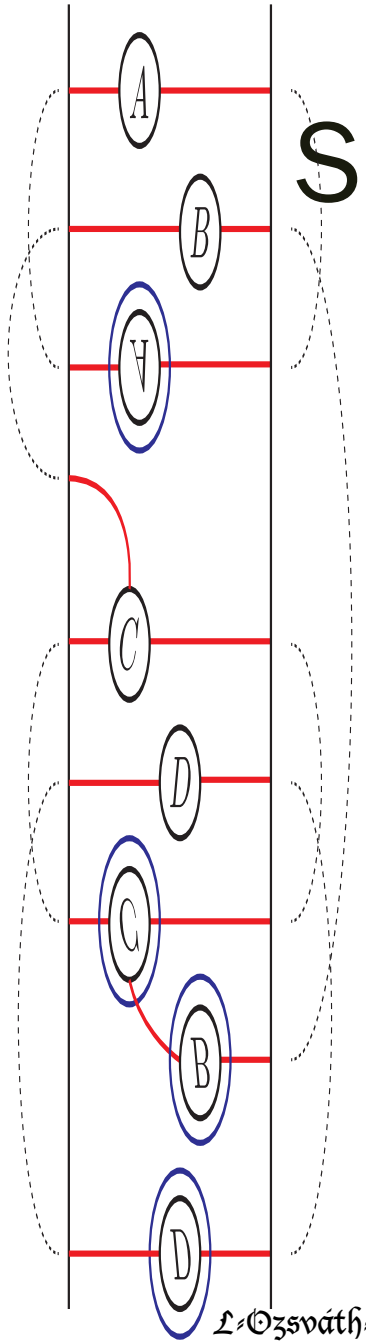

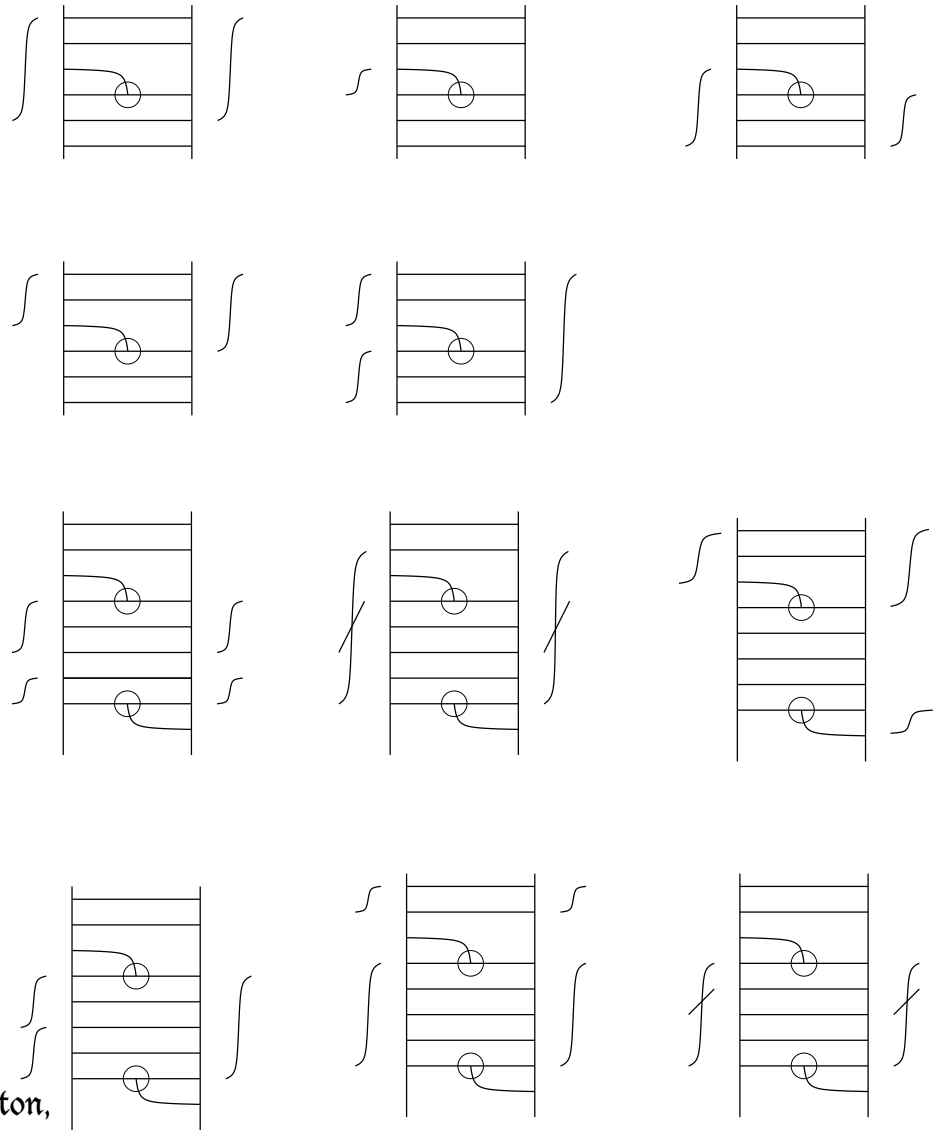

L-Ozsváth-Thurston, arXiv:1010.2550

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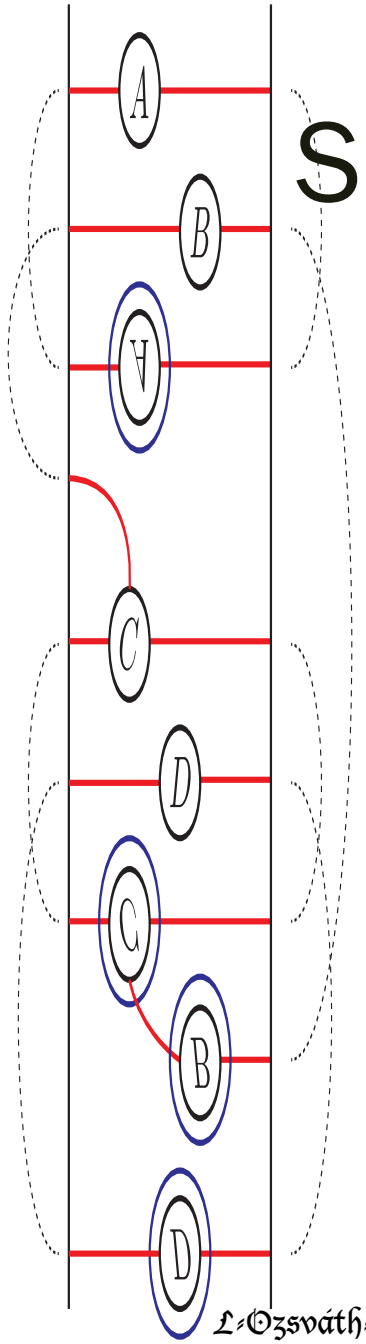

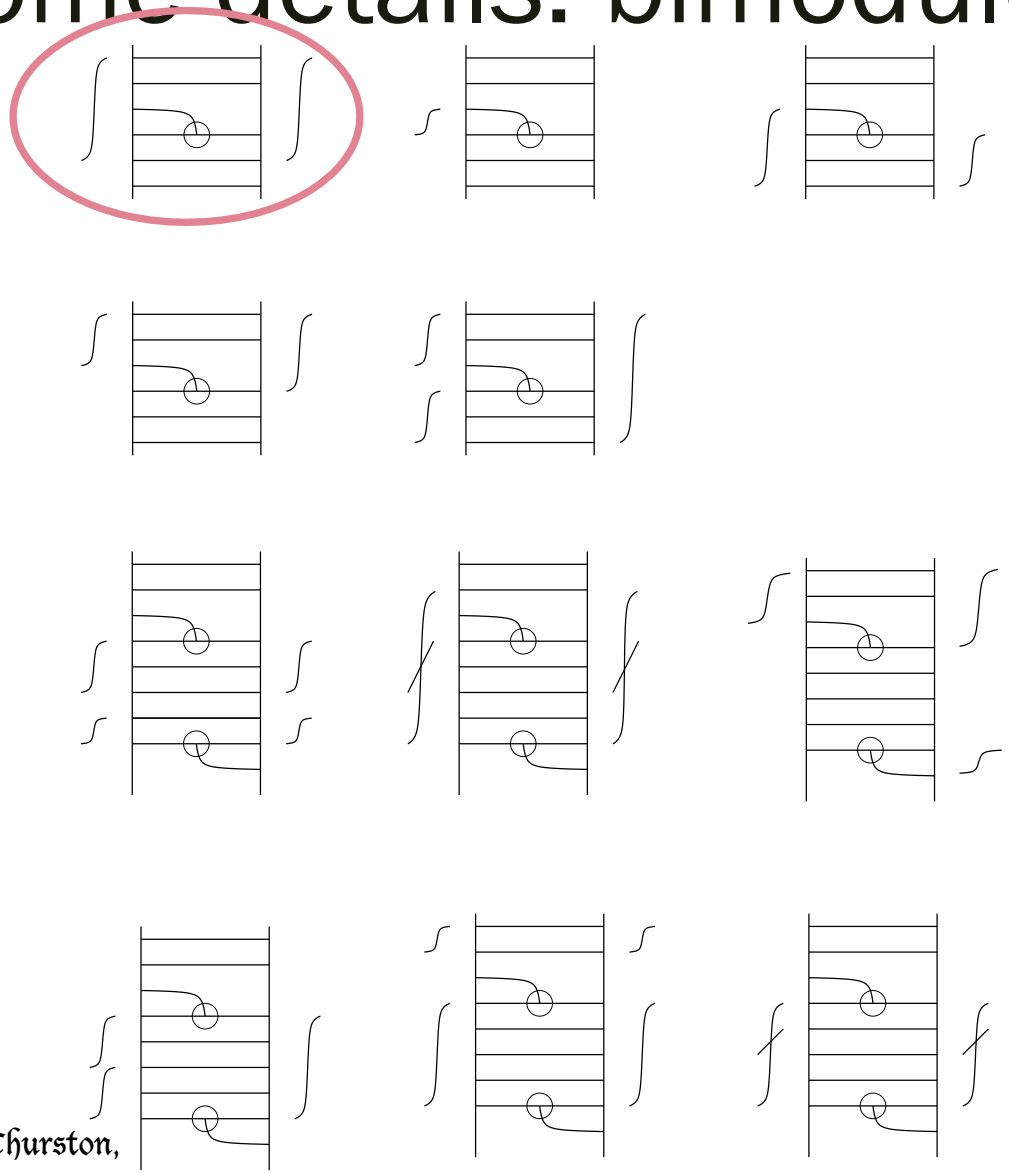

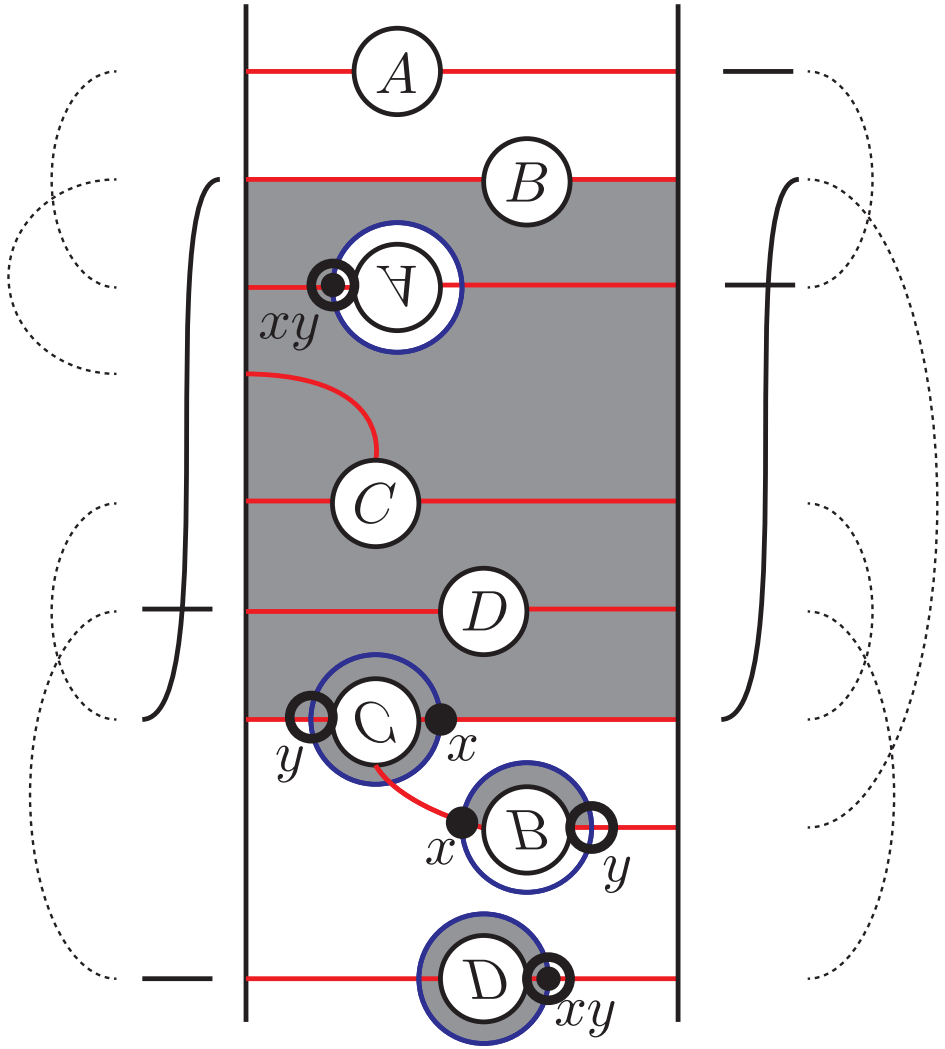

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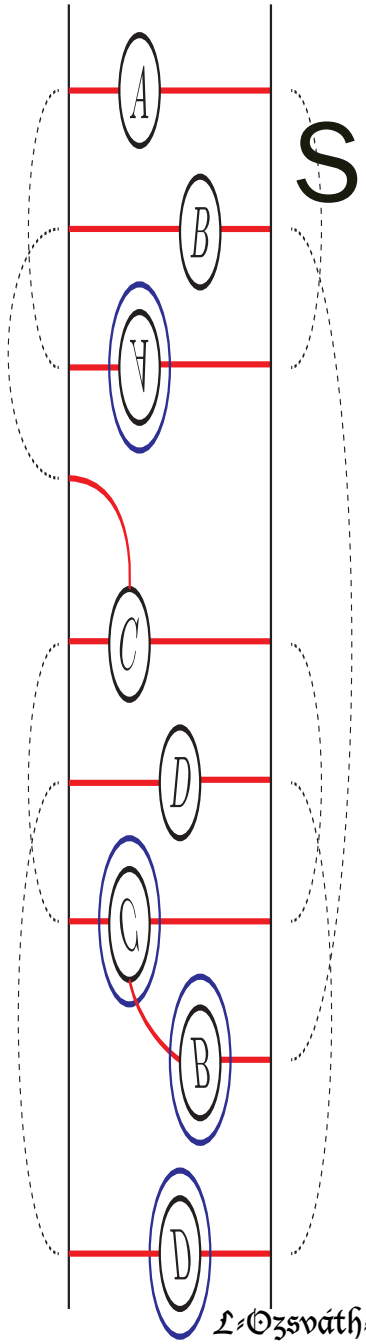

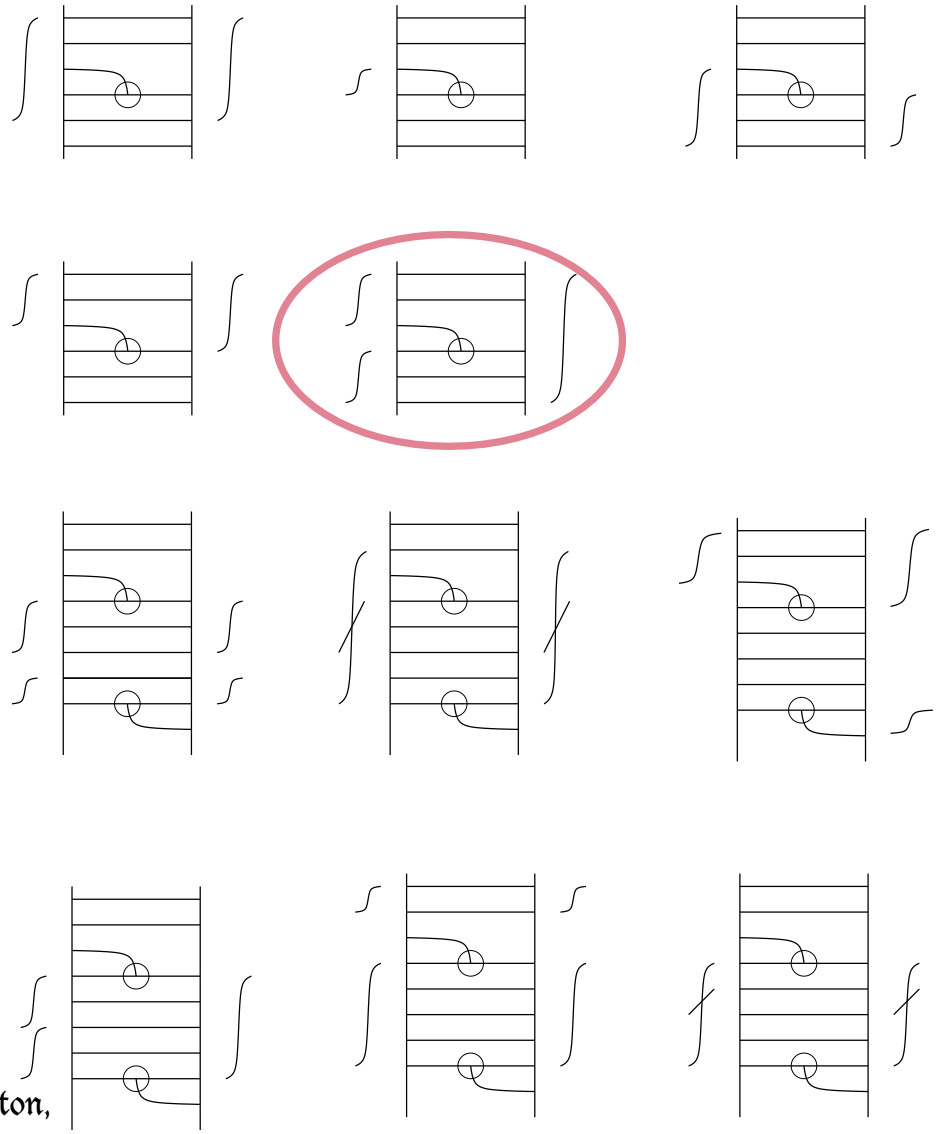

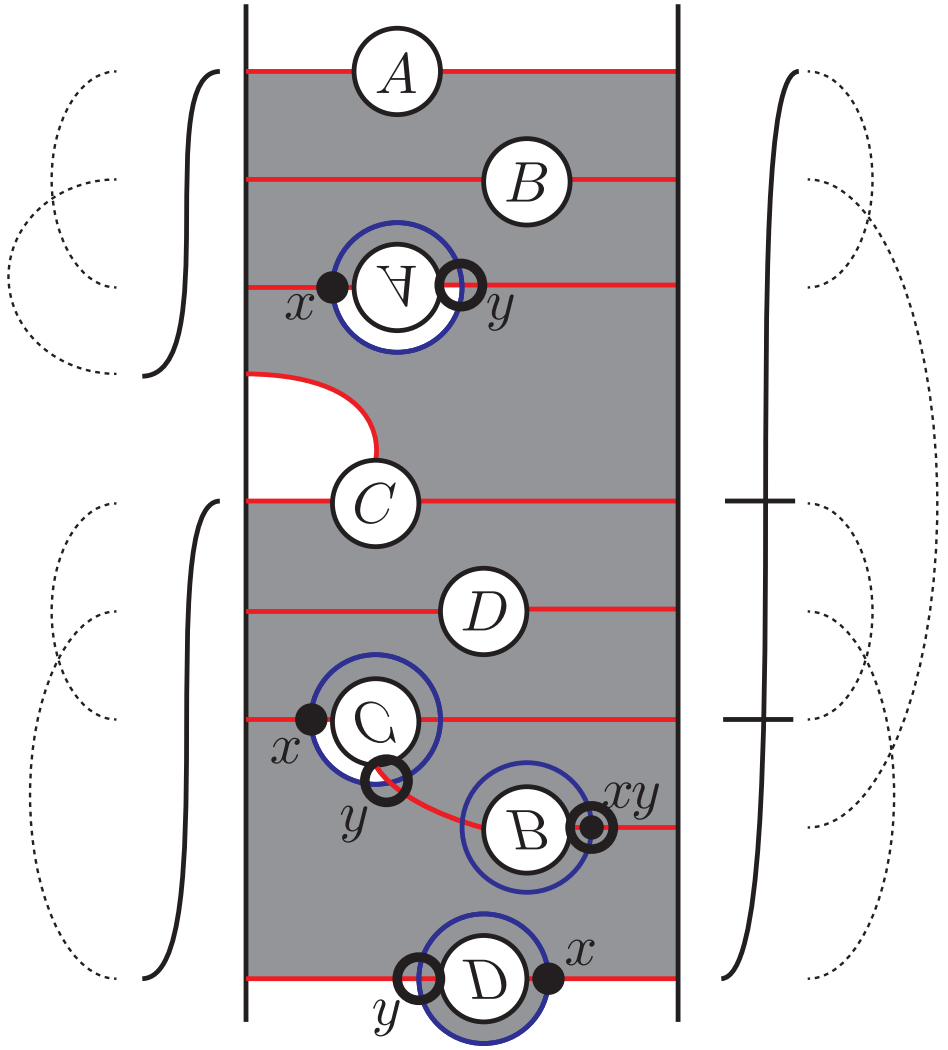

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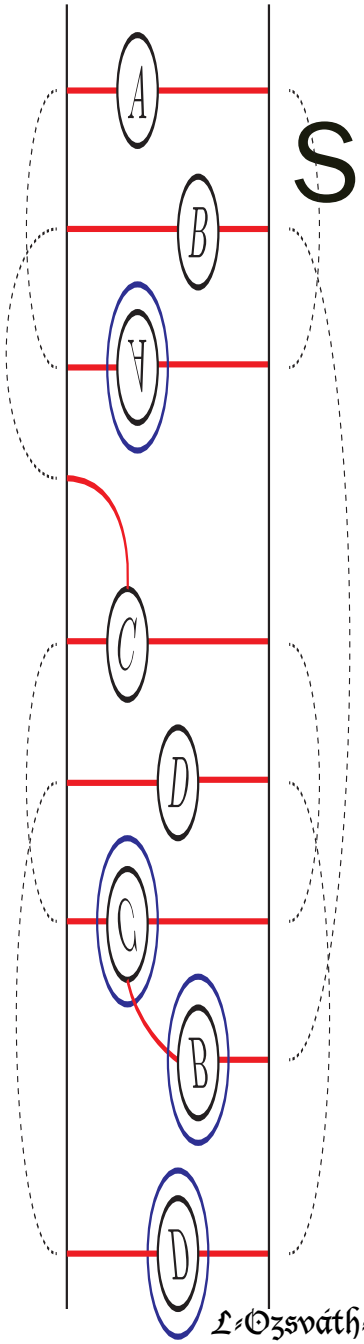
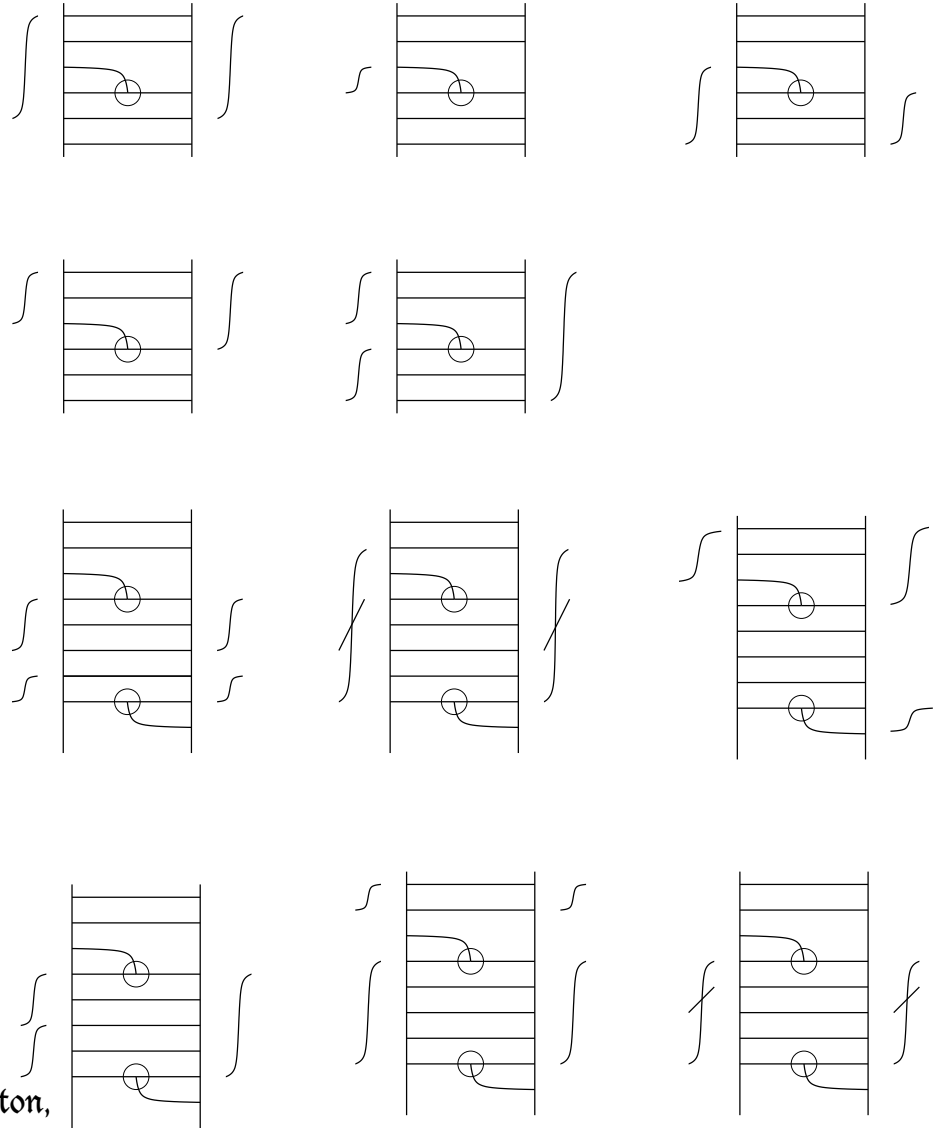
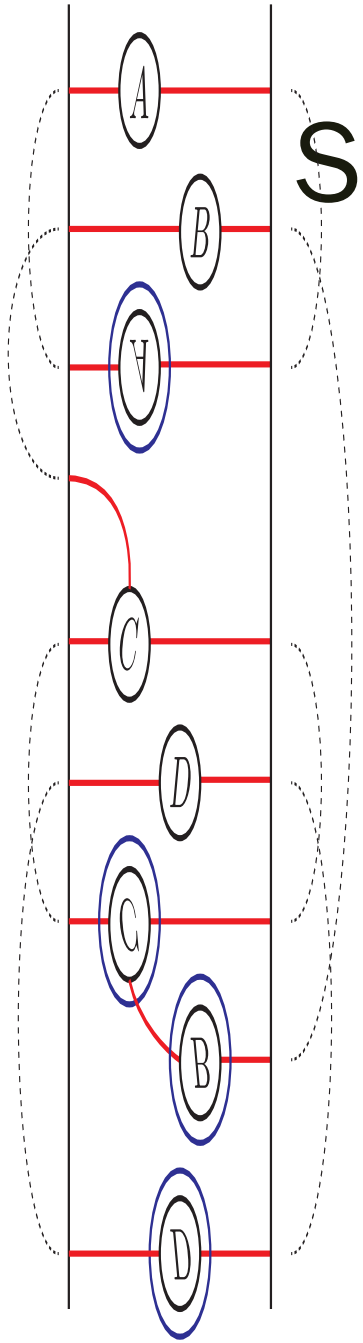




```
def _U3Chords(self):
    b1, c1, c2 = self.b1, self.c1, self.c2
    if b1 < c1:
        for x in range(c1+1, self.n):
            self._addPair([(b1, x)], [(c1, x)])
        for x in range(0, c2):
            self._addPair([(x, c2)], [(x, b1)])
    if b1 > c1:
        for x in range(c2+1, self.n):
            self._addPair([(c2, x)], [(b1, x)])
        for x in range(0, c1):
            self._addPair([(x, b1)], [(x, c1)])

def _U4Chords(self):
    b1, c1, c2 = self.b1, self.c1, self.c2
    # Two connected chords
    if b1 < c1:
        for x in range(0, b1):
            self._addPair([(x, b1)], [(x, c1)])
        for x in range(c2+1, self.n):
            if x != b1:
                self._addPair([(c2, x)], [(b1, x)])
    if b1 > c1:
        for x in range(b1+1, self.n):
            self._addPair([(b1, x)], [(c1, x)])
        for x in range(0, c2):
            if x != b1:
                self._addPair([(x, c2)], [(x, b1)])
    # Three connected chords
    if b1 < c1:
        for x in range(0, b1):
            for y in range(c1+1, self.n):
                self._addPair([(x, b1), (c1, y)], [(x, y)])
        for x in range(0, c2):
            for y in range(c2+1, self.n):
                if y != b1:
                    self._addPair([(x, y)], [(x, c2), (b1, y)])
    if b1 > c1:
        for x in range(0, c1):
            for y in range(b1+1, self.n):
                self._addPair([(x, c1), (b1, y)], [(x, y)])
        for x in range(0, c2):
            for y in range(c2+1, self.n):
                if x != b1:
                    self._addPair([(x, y)], [(x, b1), (c2, y)])

def _U5Chords(self):
    b1, c1, c2 = self.b1, self.c1, self.c2
    sc, bc = min(c1, c2), max(c1, c2)
    for x in range(0, sc):
        for y in range(bc+1, self.n):
            self._addPair([(x, sc), (bc, y)], [(x, sc), (bc, y)])

def _U6Chords(self):
    b1, c1, c2 = self.b1, self.c1, self.c2
    if b1 < c1:
        for x in range(c2+1, self.n):
            if x != b1 and x != c1:
                self._addPair([(c2, x), (b1, c1)], [(b1, x)])
        for x in range(0, b1):
            if x != c2:
                self._addPair([(x, b1)], [(x, c1), (c2, b1)])
    if b1 > c1:
        for x in range(0, c2):
            if x != b1 and x != c1:
```

L-Ozsváth-Thurston, arXiv:1010.2550

Zhan, bfh_python

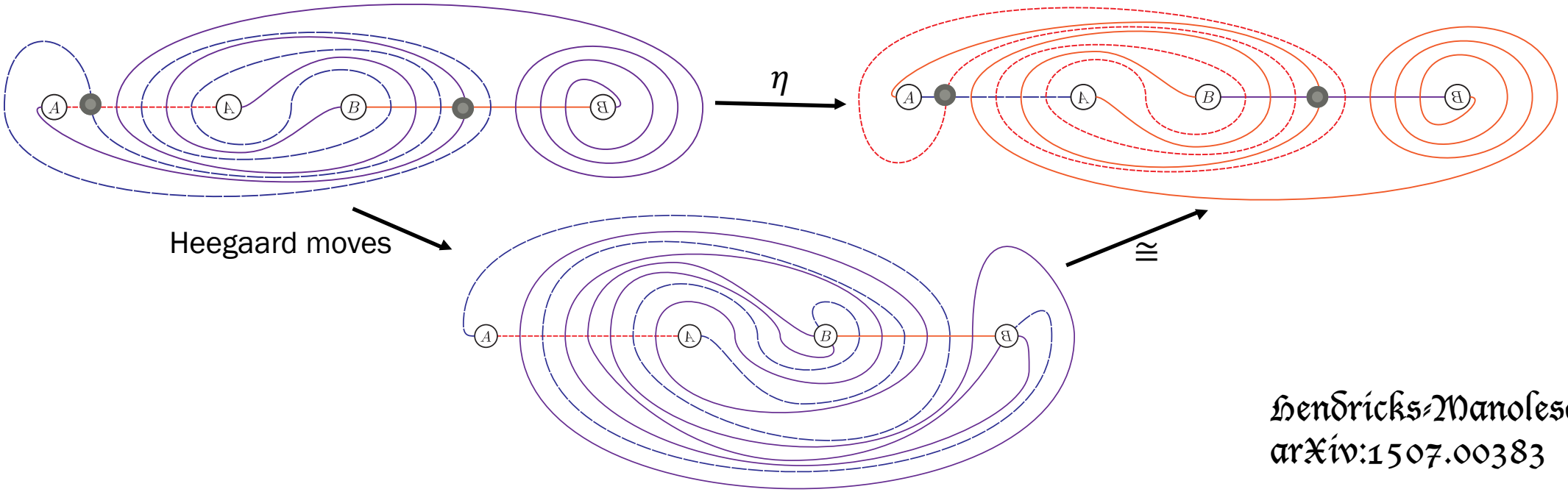


Some details: bimodules for arc slides

Zhan, bfh_python

Involutive Floer homology: definition

- Given Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ for Y, $(-\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z)$ also a diagram for Y.
- There is an obvious (*conjugation*) isomorphism $\eta\colon \widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z) \xrightarrow{\cong} \widehat{CF}(-\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z)$.
- **Theorem.** (Ozsváth-Szabó, Juhász-Thurston-Zemke, Hendricks-Manolescu) A sequence of Heegaard moves from $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ to $(-\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z)$ gives a(nother) canonical-up-to-homotopy homotopy equivalence $\Phi\colon \widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z) \xrightarrow{\simeq} \widehat{CF}(-\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z)$.



Hendricks-Manolescu, arXiv:1507.00383

Involutive Floer homology: definition

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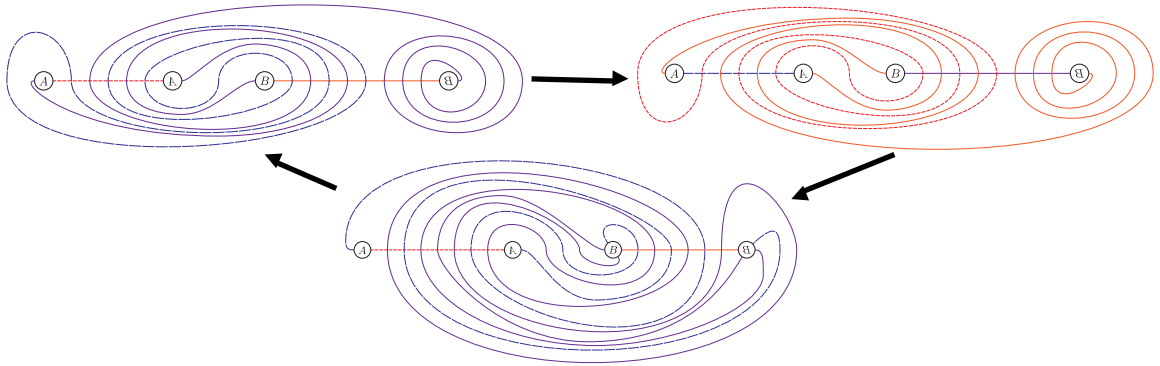
- Let $\iota := \Phi^{-1} \circ \eta\colon \widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z) \xrightarrow{\simeq} \widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$.

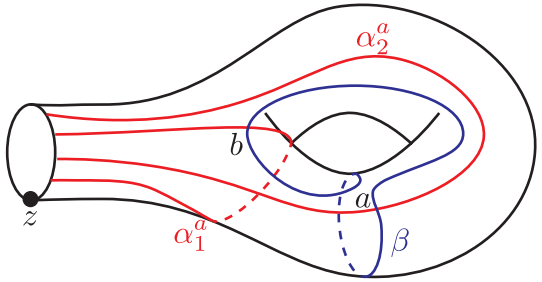
$$\widehat{CFI}(Y) = Cone(Id + \iota) = Tot\left(\widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z) \xrightarrow{Id+\iota} \widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)\right).$$

- Has an action of $F_2[Q]/(Q^2)$. There are also versions $CFI^{\pm}(Y)$ using $CF^{\pm}$.



Hendricks-Manolescu, arXiv:1507.00383

β-bordered diagrams and mirrors

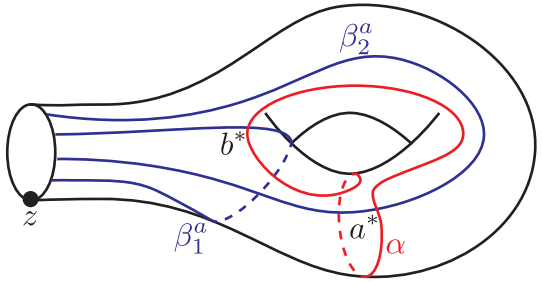


$\mathcal{H}$

$\widehat{CFD}(\mathcal{H})$

(A type D module)

$\delta^1(a) = (\rho_1 + \rho_3)b$
$\delta^1(b) = 0.$

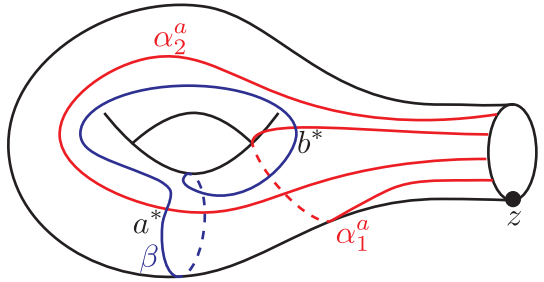


$\mathcal{H}^\beta$

$\overline{\widehat{CFD}(\mathcal{H})}$

(Its dual)

$\delta^1(b^*) = a^*(\rho_1 + \rho_3)$
$\delta^1(a^*) = 0.$

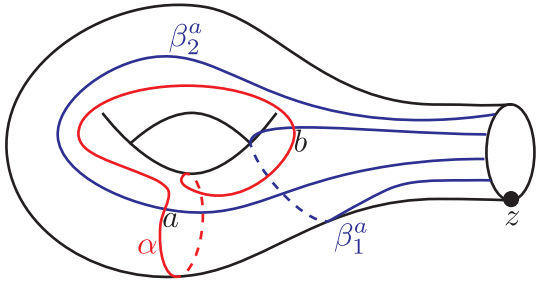


$-\mathcal{H}$

$\overline{\widehat{CFD}(\mathcal{H})}$

(Also its dual)

$\delta^1(b^*) = a^*(\rho_1 + \rho_3)$
$\delta^1(a^*) = 0.$



$-\mathcal{H}^\beta$

$\widehat{CFD}(\mathcal{H})$

(The original module)

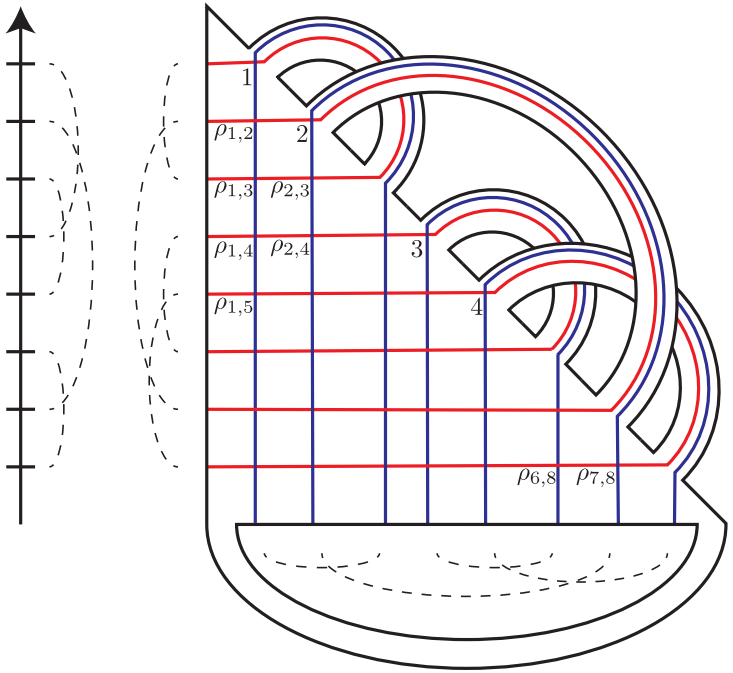
$\delta^1(a) = (\rho_1 + \rho_3)b$
$\delta^1(b) = 0.$

Similarly for $\widehat{CFA}$.

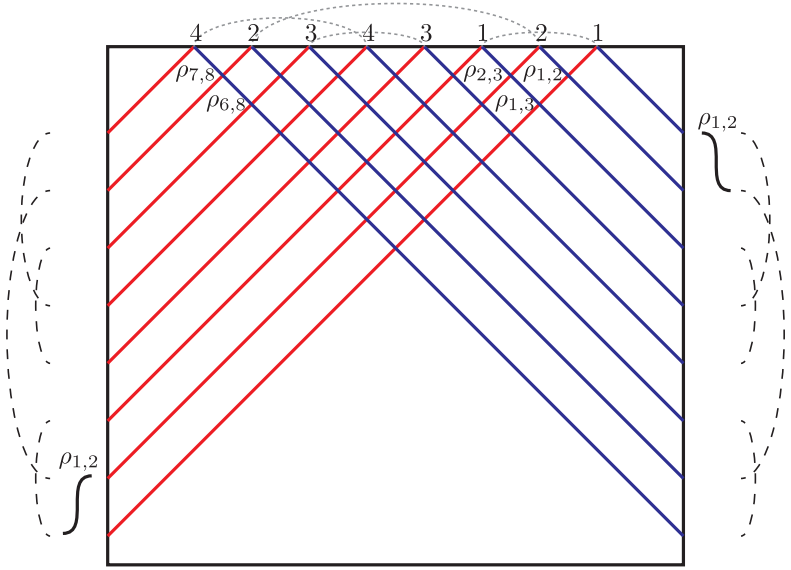
L-Ozsváth-Thurston, arXiv:1005.1248

The Auroux-Zarev diagram

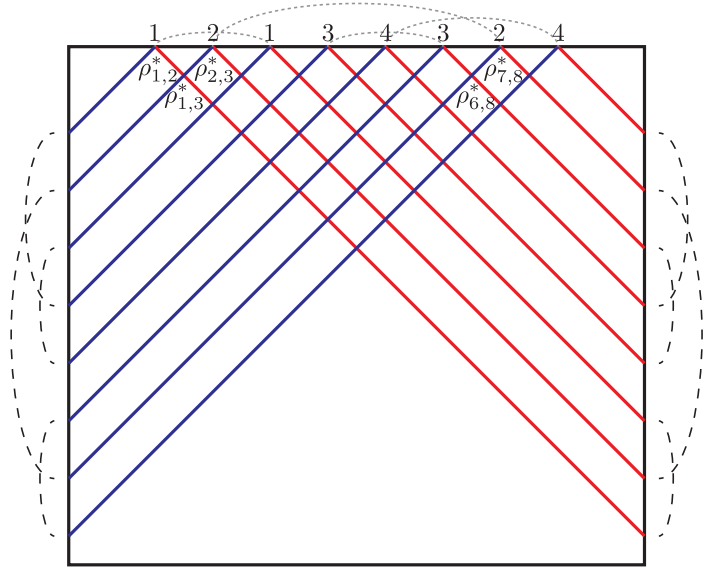
- A miraculous pair of α-β-bordered diagrams $AZ(\mathcal{Z})$, $\overline{AZ}(\mathcal{Z})$ associated to a pointed matched circle $\mathcal{Z}$.



$AZ(\mathcal{Z})$



$AZ(\mathcal{Z})$



$\overline{AZ}(-\mathcal{Z})$

Auroux, arXiv:1001.4323
Zarev, arXiv:1010.3496

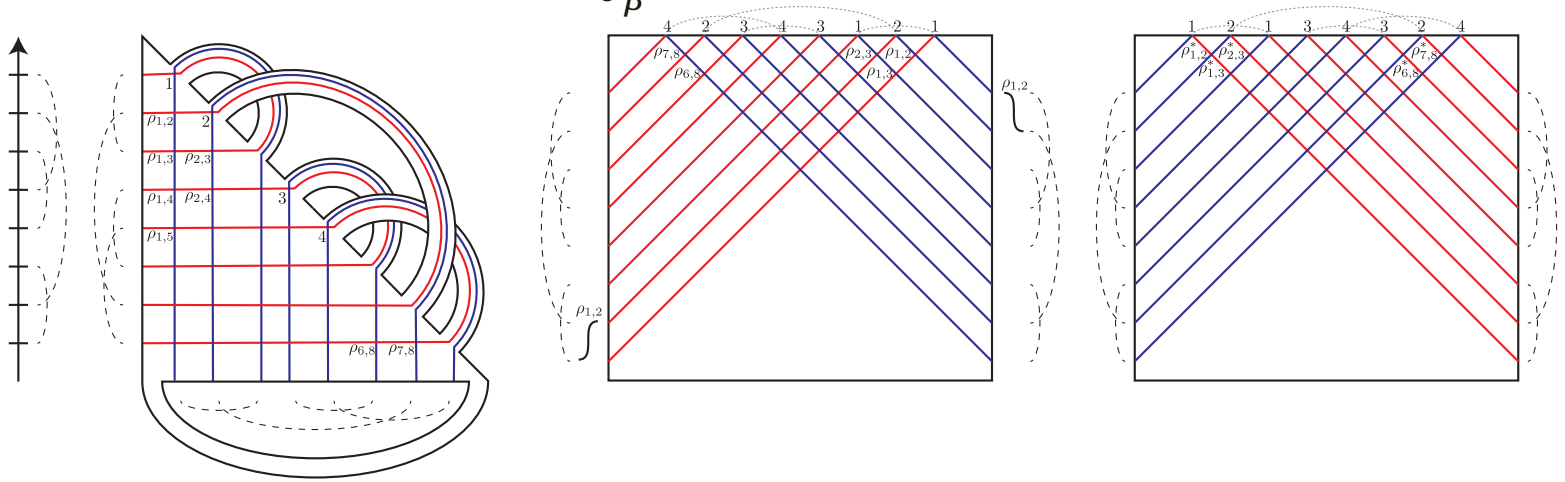
L-Ozsváth-Thurston, arXiv:1005.1248

The Auroux-Zarev diagram

- A miraculous pair of α-β-bordered diagrams $AZ(\mathcal{Z})$, $\overline{AZ}(\mathcal{Z})$ associated to a pointed matched circle $\mathcal{Z}$.
- **Theorem.** (Auroux, Zarev) $\widehat{CFAA}\big(AZ(\mathcal{Z})\big) \cong \mathcal{A}(\mathcal{Z})$.
- $\widehat{CFDA}\big(AZ(\mathcal{Z})\big)$, $\widehat{CFDA}\big(\overline{AZ}(\mathcal{Z})\big)$ easy to understand.
- **Lemma.** (LOT) For any bordered Heegaard diagram $\mathcal{H}$ with one boundary component,

$$-\mathcal{H}^{\beta} \cup_{\partial} AZ(\mathcal{Z}) \sim \mathcal{H} \sim -\mathcal{H}^{\beta} \cup_{\partial} \overline{AZ}(\mathcal{Z}).$$

- **Lemma.** (LOT) $\overline{AZ}(\mathcal{Z}) \cup_{\partial_\beta} AZ(-\mathcal{Z}) \sim Id$.



Auroux, arXiv:1001.4323
Zarev, arXiv:1010.3496

L-Ozsváth-Thurston, arXiv:1005.1248

Some rigidity results

- **Lemma**. Let Y be a handlebody with boundary $F(-\mathcal{Z})$ and M an $\mathcal{A}(\mathcal{Z})$-module homotopy equivalent to $\widehat{CFD}(Y)$. Then, up to homotopy, there is a *unique* graded homotopy equivalence $M \simeq \widehat{CFD}(Y)$.
- A similar statement holds for $\widehat{CFA}(Y)$.
- **Lemma.** Let Id be the identity cobordism of $F(\mathcal{Z})$ and M an $\mathcal{A}(\mathcal{Z})$-bimodules homotopy equivalent to $\widehat{CFDA}(Id)$. Then up to homotopy, there is a *unique* graded homotopy equivalence $M \simeq \widehat{CFDA}(Id)$.
- So, given a module M homotopy equivalent to $\widehat{CFD}(Y)$, can compute *the* homotopy equivalence $M \overset{\simeq}{\leftrightarrow} \widehat{CFD}(Y)$.

L-Ozsváth-Thurston, Unpublished draft Hendricks-L

Proof of rigidity

Lemma. Let Y be a handlebody with boundary $F(-\mathcal{Z})$ and M an $\mathcal{A}(\mathcal{Z})$-module homotopy equivalent to $\widehat{CFD}(Y)$. Then, up to homotopy, there is a *unique* graded homotopy equivalence $M \simeq \widehat{CFD}(Y)$.

Proof.

- Let Y_0 be the 0-framed handlebody.
- $Y_0 = \phi(Y) = Y \cup_{\text{bdy}} Y_\phi$ for some $\phi \in MCG$.
- $\widehat{CFD}(Y_0) = \widehat{CFDA}(\phi) \otimes \widehat{CFD}(Y)$.
- $\widehat{CFDA}(\phi) \otimes \cdot$ is an equivalence of categories, with inverse $\widehat{CFDA}(\phi^{-1}) \otimes \cdot$
- So, $HE\left(\widehat{CFD}(Y), M\right) \leftrightarrow HE(\widehat{CFD}(Y_0), \widehat{CFDA}(\phi) \otimes M)$.
- $HE\left(\widehat{CFD}(Y_0), \widehat{CFDA}(\phi) \otimes M\right) \leftrightarrow HE\left(\widehat{CFD}(Y_0), \widehat{CFD}(Y_0)\right)$.
- $HE\left(\widehat{CFD}(Y_0), \widehat{CFD}(Y_0)\right)$ has one element, by inspection.

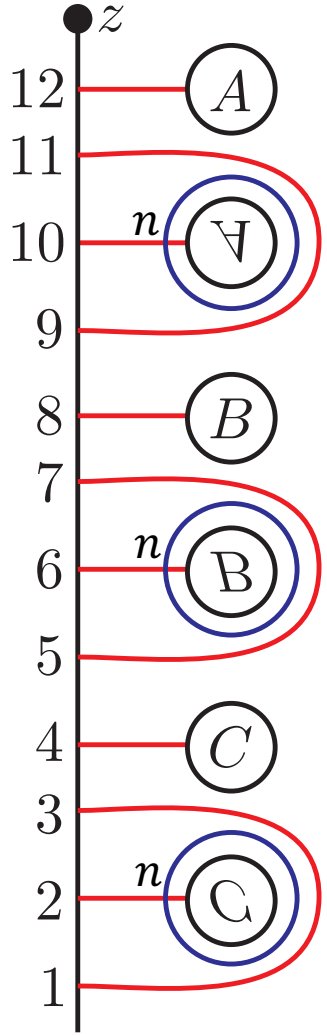
Let $HE(M, N)$ be the set of graded homotopy equivalences from M to N.

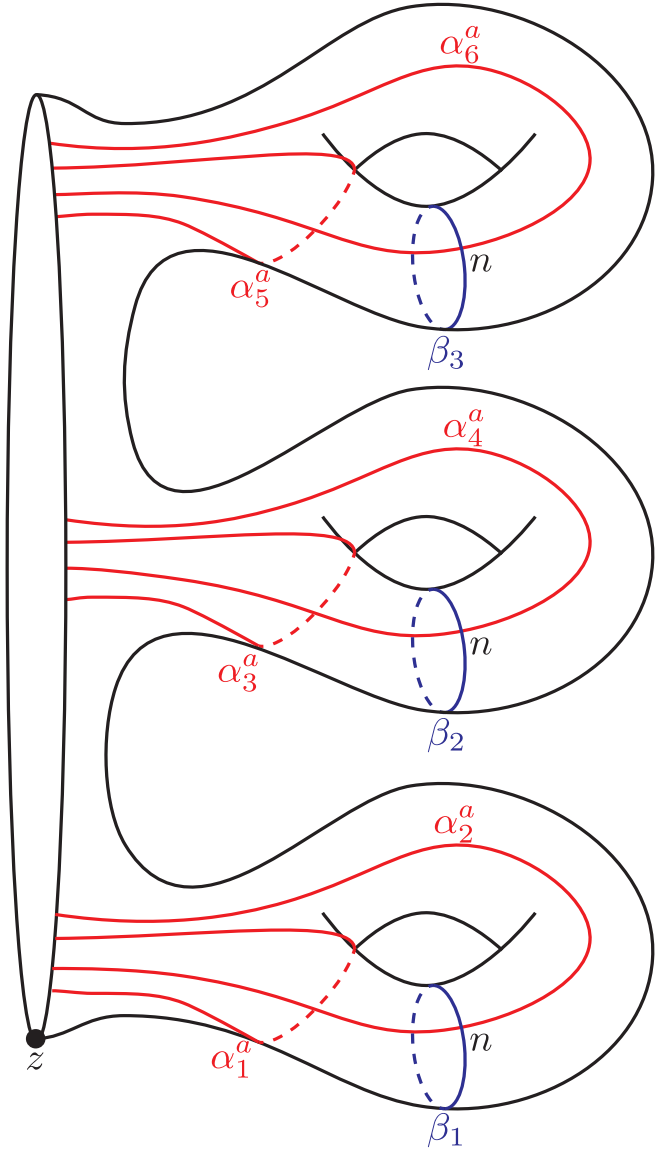
L-Ozsváth-Thurston, Unpublished draft
Hendricks-L

[Recall: modules for 0-framed handlebodies]

$$\widehat{CFD}(Y_0) = \mathcal{A}(\mathcal{Z})\langle n \rangle$$

$$\partial(n) = (\rho_{1,3} + \rho_{5,7} + \rho_{9.11} + \cdots)n$$





Computing involutive Floer homology

- Fix a Heegaard splitting $Y = Y_0 \cup Y_1$, corresponding bordered Heegaard diagrams $\mathcal{H}_0, \mathcal{H}_1$.
- Since $\overline{AZ} \cup AZ$ represents $Id_{\mathcal{Z}}$, $\exists\, \Phi$: $\widehat{CFDA}(\mathrm{Id}) \xrightarrow{\simeq} \widehat{CFDA}(\overline{AZ}) \otimes \widehat{CFDA}(AZ)$.
  - *By rigidity, this equivalence is unique.*
- Since $\mathcal{H}_0$ and $-\mathcal{H}_0^{\beta} \cup_{\partial} \overline{AZ}$ both represent Y_0,

  $\exists\, \Psi_A$: $\widehat{CFA}(\mathcal{H}_0) \otimes_{\mathcal{A}(\mathcal{Z})} \widehat{CFDA}(\overline{AZ}) = \widehat{CFA}\left(-\mathcal{H}_0^{\beta}\right) \otimes_{\mathcal{A}(\mathcal{Z})} \widehat{CFDA}(\overline{AZ}) \xrightarrow{\simeq} \widehat{CFA}(\mathcal{H}_0)$.

  Since $\mathcal{H}_1$ and $AZ \cup_{\partial} \left(-\mathcal{H}_1^{\beta}\right)$ both represent Y_1,

  $\exists\, \Psi_D$: $\widehat{CFD}(\mathcal{H}_1) \otimes_{\mathcal{A}(\mathcal{Z})} \widehat{CFDA}(\overline{AZ}) = \widehat{CFD}\left(-\mathcal{H}_1^{\beta}\right) \otimes_{\mathcal{A}(\mathcal{Z})} \widehat{CFDA}(\overline{AZ}) \xrightarrow{\simeq} \widehat{CFA}(\mathcal{H}_1)$.

- **Theorem.** The map ι is the composition

$$\widehat{CFA}(\mathcal{H}_0) \otimes_{\mathcal{A}(\mathcal{Z})} \widehat{CFD}(\mathcal{H}_1) = \widehat{CFA}(\mathcal{H}_0) \otimes_{\mathcal{A}(\mathcal{Z})} \widehat{CFDA}(Id) \otimes_{\mathcal{A}(\mathcal{Z})} \widehat{CFD}(\mathcal{H}_1)$$
$$\xrightarrow{Id \otimes \Phi \otimes Id} \widehat{CFA}(\mathcal{H}_0) \otimes_{\mathcal{A}(\mathcal{Z})} \widehat{CFDA}(\overline{AZ}) \otimes_{\mathcal{A}(\mathcal{Z})} \widehat{CFDA}(AZ) \otimes_{\mathcal{A}(\mathcal{Z})} \widehat{CFD}(\mathcal{H}_1)$$
$$\xrightarrow{\Psi_A \otimes \Psi_D} \widehat{CFA}(\mathcal{H}_0) \otimes_{\mathcal{A}(\mathcal{Z})} \widehat{CFD}(\mathcal{H}_1).$$

Proof that this works

Theorem. The map ι is the composition

$$\begin{aligned}\widehat{CFA}(\mathcal{H}_0) \otimes_{\mathcal{A}(\mathcal{Z})} \widehat{CFD}(\mathcal{H}_1) &= \widehat{CFA}(\mathcal{H}_0) \otimes_{\mathcal{A}(\mathcal{Z})} \widehat{CFDA}(Id) \otimes_{\mathcal{A}(\mathcal{Z})} \widehat{CFD}(\mathcal{H}_1) \\ &= \widehat{CFA}\left(-\mathcal{H}_0^{\beta}\right) \otimes_{\mathcal{A}(\mathcal{Z})} \widehat{CFDA}(Id) \otimes_{\mathcal{A}(\mathcal{Z})} \widehat{CFD}\left(-\mathcal{H}_1^{\beta}\right) \\ &\xrightarrow{Id\otimes\Phi\otimes Id} \widehat{CFA}\left(-\mathcal{H}_0^{\beta}\right) \otimes_{\mathcal{A}(\mathcal{Z})} \widehat{CFDA}(\overline{AZ}) \otimes_{\mathcal{A}(\mathcal{Z})} \widehat{CFDA}(AZ) \otimes_{\mathcal{A}(\mathcal{Z})} \widehat{CFD}\left(-\mathcal{H}_1^{\beta}\right) \\ &\xrightarrow{\Psi_A\otimes\Psi_D} \widehat{CFA}(\mathcal{H}_0) \otimes_{\mathcal{A}(\mathcal{Z})} \widehat{CFD}(\mathcal{H}_1).\end{aligned}$$

Proof. Starting with the closed diagram $\mathcal{H}_0 \cup \mathcal{H}_1$:

- The first equality is a sequence of Heegaard moves.
- The second equality is the obvious isomorphism exchanging α- and β-circles and switching the orientation.
- There are sequences of Heegaard moves given isomorphisms as in the next two equalities.
  - *By the pairing theorem for triangles, these have the form $Id \otimes \Phi \otimes Id$ and $\Psi_A \otimes \Psi_D$ for some Φ, Ψ_A, Ψ_D.*
  - *Rigidity implies these maps are unique.*

Is this helpful?

- Bohua Zhan's implementation (`github.com/bzhan/bfh_python`) of the bordered algorithm for computing $\widehat{HF}(Y)$ can compute interesting examples.
- We're working to implement computation of $\widehat{HFI}(Y)$.
- Expect it to be fast enough for computations, but
- Most applications of involutive Floer homology use $HFI^{\pm}(Y)$, but
- Sometimes one can deduce $HFI^{\pm}(Y)$ from $\widehat{HFI}(Y)$ and $HF^{\pm}(Y)$.

Involutive bordered Floer homology: definition

- This slide was wrong when I gave the talk. Please see our paper arXiv:1706.06557, instead.

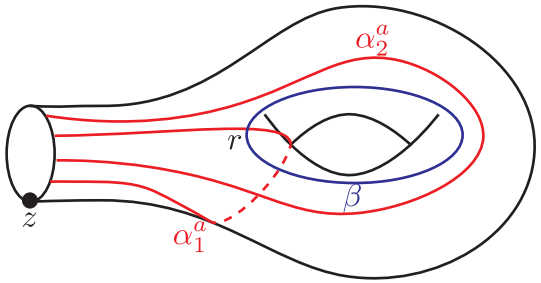
Involutive bordered Floer homology: structure and questions

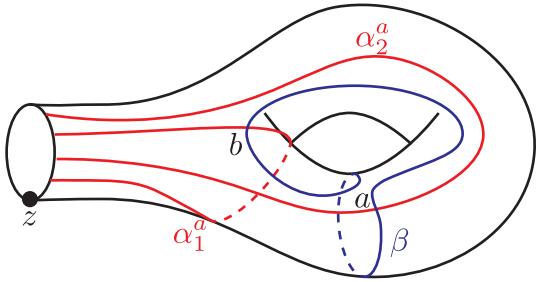
- **Theorem** (Pairing Theorem). Given bordered Y_1, Y_2, with $\partial Y_1 = F(\mathcal{Z}) = -\partial Y_2$
$$\widehat{CFI}(Y_1 \cup_\partial Y_2) \simeq \widehat{CFAI}(Y_1) \otimes \quad \widehat{CFDI}(Y_2).$$
- Recall that defining $\widehat{CFAI}(Y_1)$ and $\widehat{CFDI}(Y_2)$ used a sequence of Heegaard moves.
- The pairing theorem holds for any choice of such a sequence.
- **Conjecture** (Invariance). The modules $CFAI(Y)$ and $CFDI(Y)$ are invariants of the bordered manifold Y, i.e., do not depend on the chosen sequence of Heegaard moves.

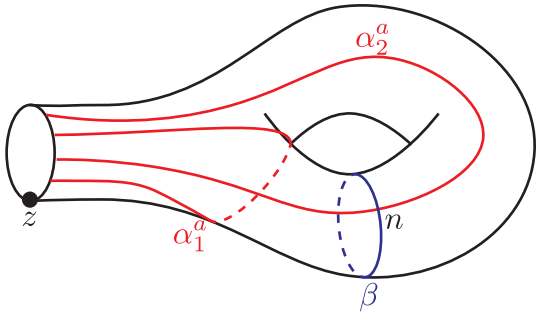
Application: surgery exact triangle

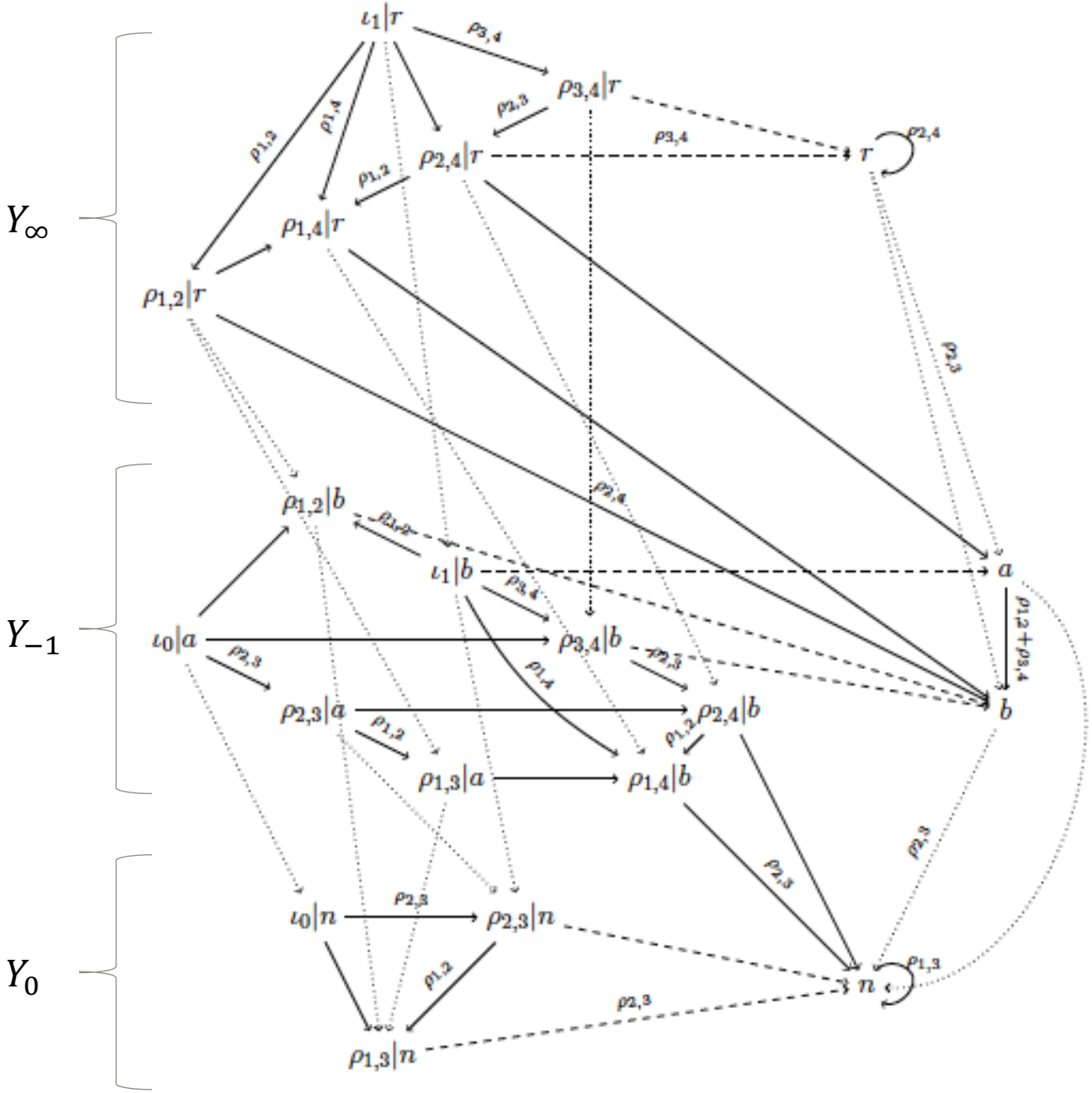
- **Theorem.** Let K be a framed knot in a 3-manifold Y. Then there is a long exact sequence
$$\cdots \to \widehat{HFI}\left(Y_{-1}(K)\right) \to \widehat{HFI}\left(Y_0(K)\right) \to \widehat{HFI}(Y) \to \widehat{HFI}\left(Y_{-1}(K)\right) \to \cdots.$$
- Proof. Let H_{-1}, H_0, and H_∞ be the -1-framed, 0-framed, and ∞-framed solid tori. Enough to find a short exact sequence
$$0 \to CFDI(H_{-1}) \to CFDI(H_0) \to CFDI(H_\infty) \to 0.$$
- Do so, by direct computation.

Direct computation









Summary

- Can compute $\widehat{HF}(Y)$ from a Heegaard splitting and factoring in the mapping class group(oid), via bordered Floer homology.
- The involutive Floer homology $\widehat{HFI}(Y)$ is then determined from the trivial-looking maps

$$Hom\left(\widehat{CFD}(H_0), \widehat{CFD}(H_1)\right) \to Hom\left(\widehat{CFDA}(AZ) \otimes \widehat{CFD}(H_0), \widehat{CFDA}(AZ) \otimes \widehat{CFD}(H_1)\right)$$
$$\to Hom\left(\widehat{CFD}(H_0), \widehat{CFD}(H_1)\right)$$

or

$$\widehat{CFA}(\mathcal{H}_0) \otimes_{\mathcal{A}(Z)} \widehat{CFD}(\mathcal{H}_1) = \widehat{CFA}(\mathcal{H}_0) \otimes_{\mathcal{A}(Z)} \widehat{CFDA}(Id) \otimes_{\mathcal{A}(Z)} \widehat{CFD}(\mathcal{H}_1)$$
$$\xrightarrow{Id \otimes \Phi \otimes Id} \widehat{CFA}(\mathcal{H}_0) \otimes_{\mathcal{A}(Z)} \widehat{CFDA}(\overline{AZ}) \otimes_{\mathcal{A}(Z)} \widehat{CFDA}(AZ) \otimes_{\mathcal{A}(Z)} \widehat{CFD}(\mathcal{H}_1)$$
$$\xrightarrow{\Psi_A \otimes \Psi_D} \widehat{CFA}(\mathcal{H}_0) \otimes_{\mathcal{A}(Z)} \widehat{CFD}(\mathcal{H}_1).$$

- This is computable because there are unique equivalences between invariants of handlebodies.