The Alexander polynomial and knot Floer homology

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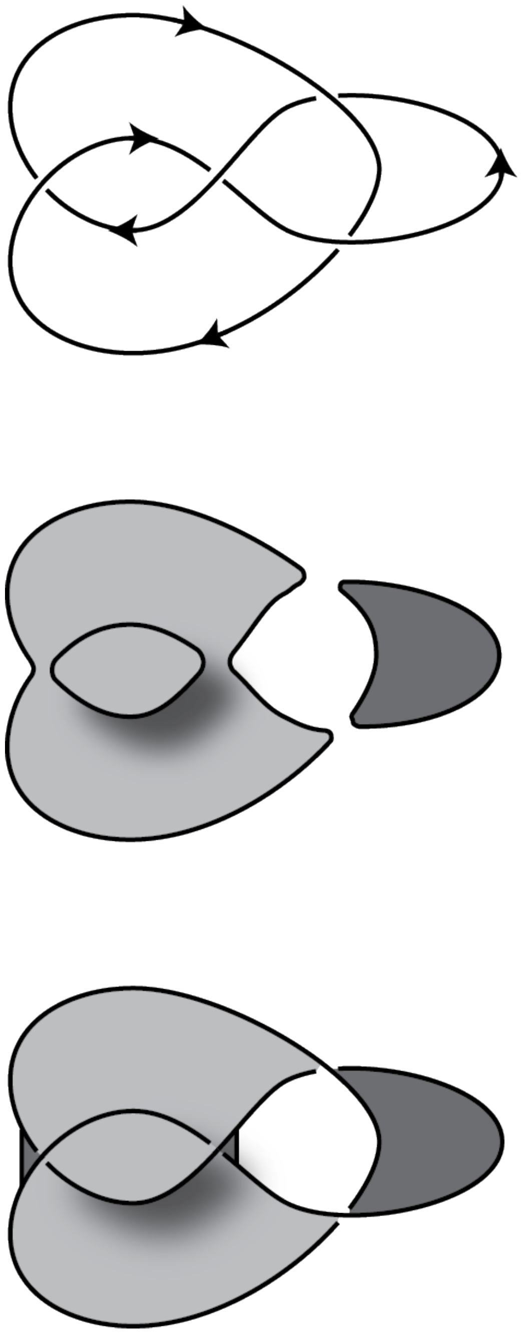
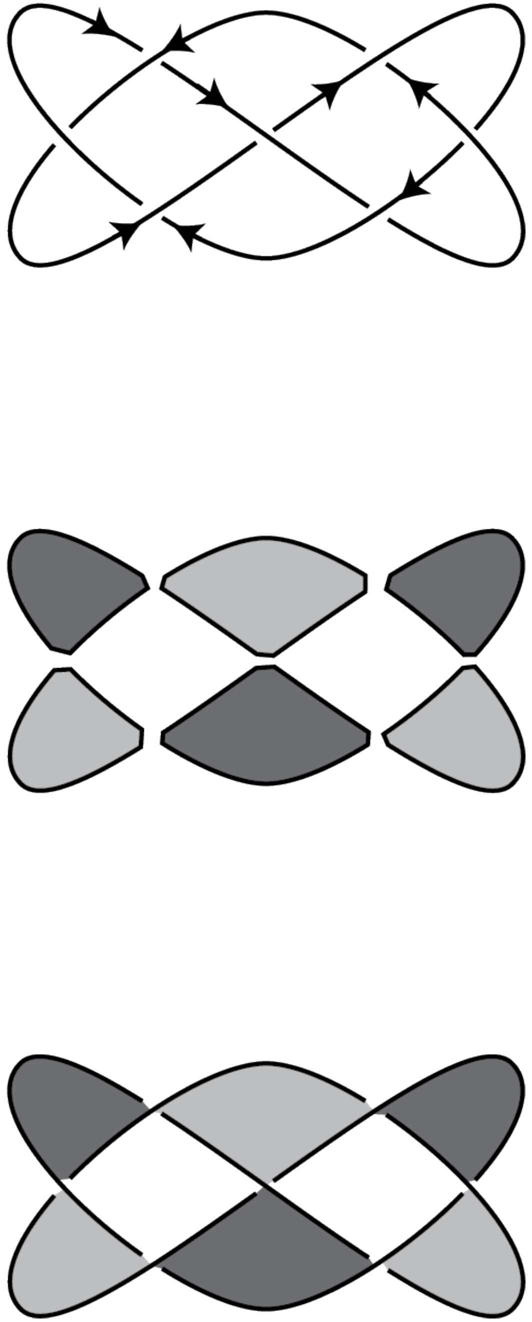
(Mostly work of other people. Partly joint with Peter Ozsváth and Dylan Thurston, or David Treumann, or Kristen Hendricks and Sucharit Sarkar.)

1. RL was supported by NSF grant DMS-1149800. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation."

Seifert surfaces

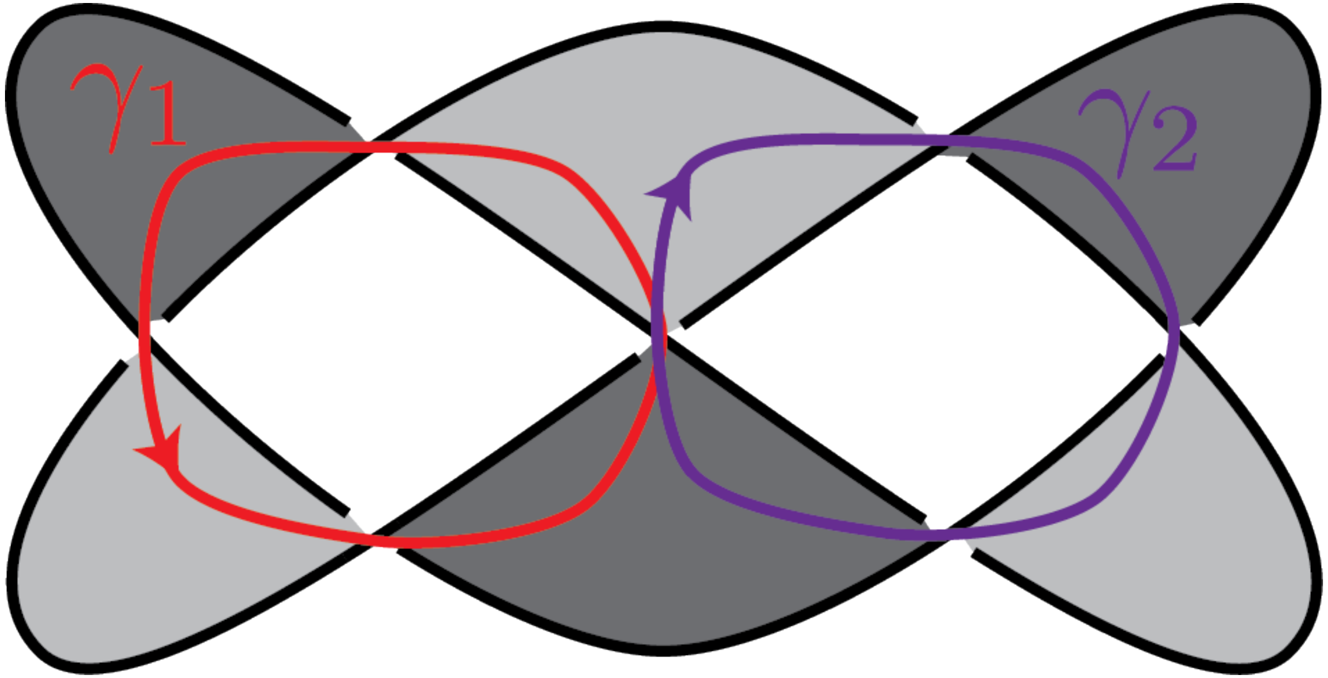
- Every knot K in S^3 bounds an orientable surface.
- **Definition**. Minimal genus of such a surface is $g(K)$.
- **Definition**. An orientable surface with boundary K is a *Seifert surface* for K.





Seifert matrix and Alexander polynomial

- F -- a Seifert surface for K.
- Let $\{\gamma_i\}_{i=1}^{2g}$ -- a basis for $H_1(F)$.



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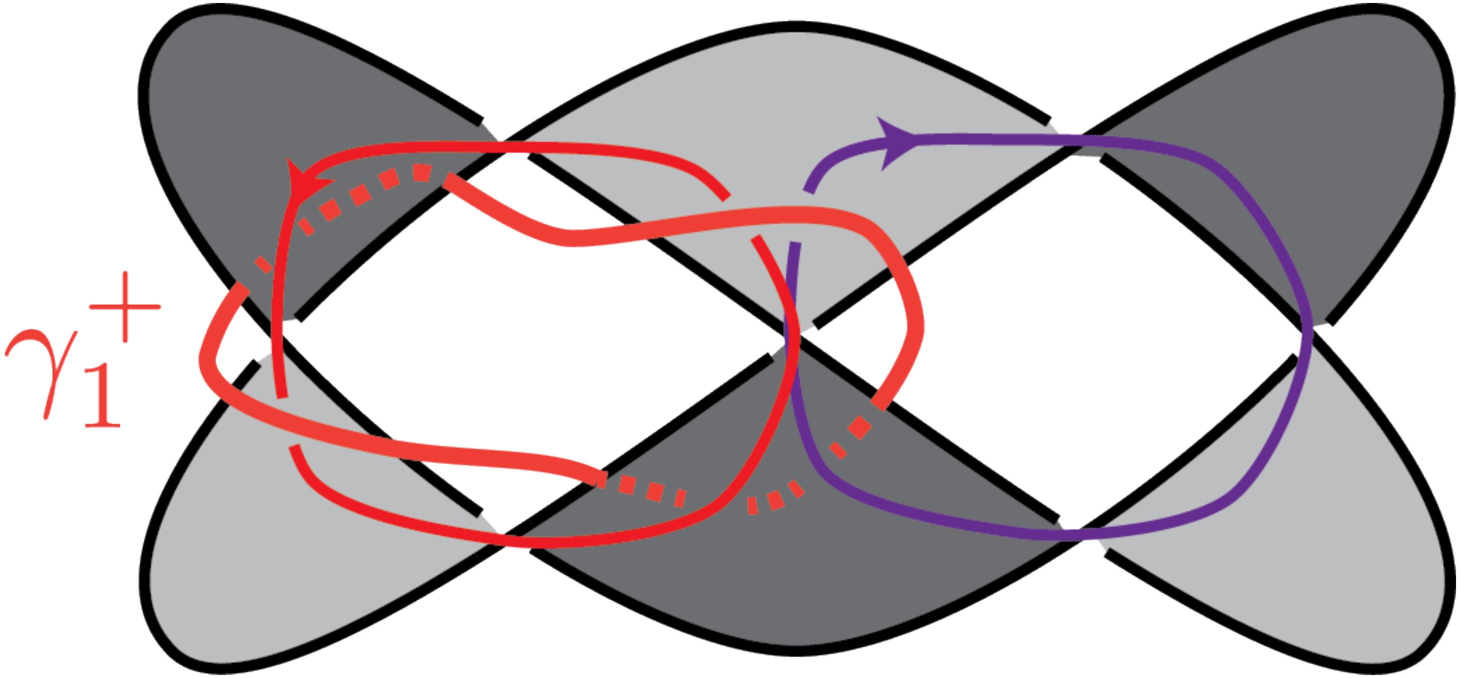
- γ_i^+ be a positive pushoff of γ_i.
- **Definition**. Seifert matrix $A = (a_{i,j})$ with
$$a_{i,j} = lk(\gamma_i, \gamma_j^+).$$
- **Example**: $\begin{pmatrix} \mathbf{-2} & \mathbf{-1} \\ 0 & -2 \end{pmatrix}$



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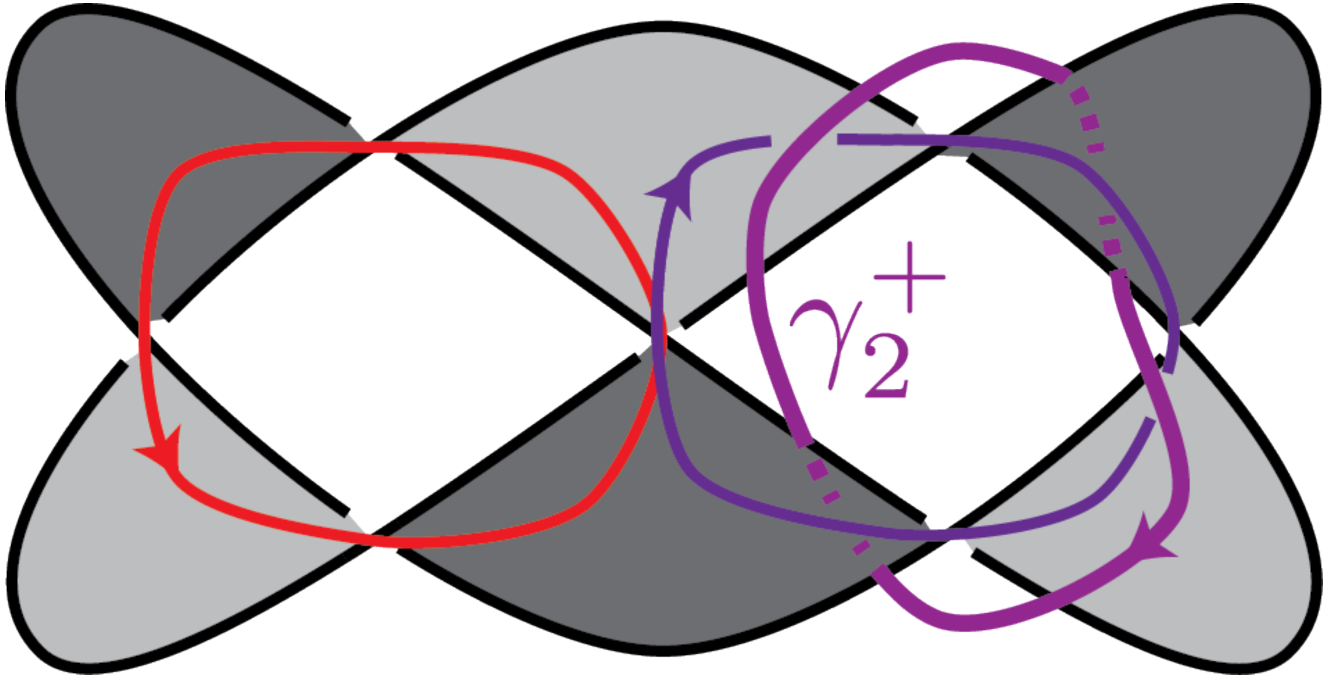

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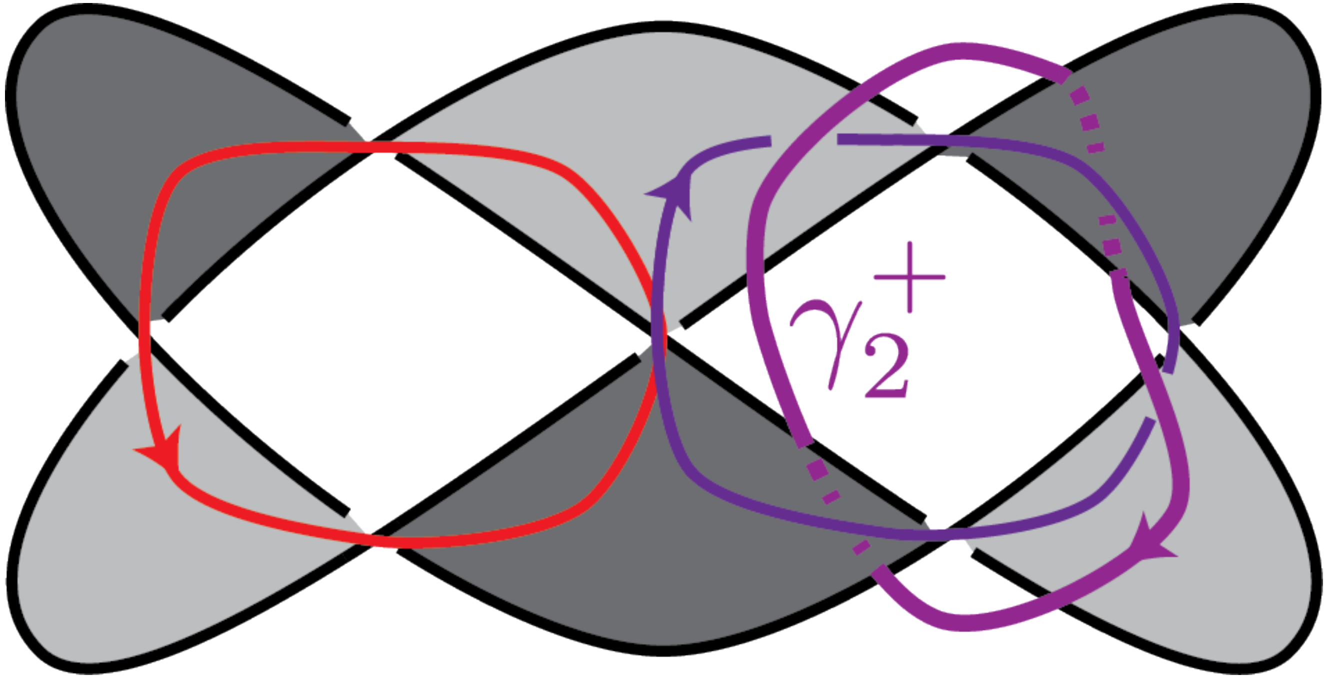
- **Example**: $\begin{pmatrix} -2 & -1 \\ 0 & -2 \end{pmatrix}$
- **Definition**. Alexander polynomial
$$\Delta_K(t) = \pm t^n \det(tA - A^T).$$
- **Example**:

$$\Delta_K(t) = \pm t^n \begin{vmatrix} -2t+2 & -t \\ 1 & -2t+2 \end{vmatrix}$$
$$= \pm t^n(-4t^2 + 7t - 4) = 4t - 7 + 4t^{-1}.$$

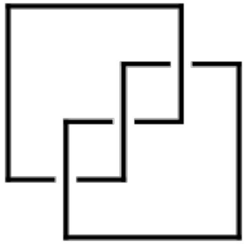
- **Corollary**. $2g(K) \geq width(\Delta_K(t))$



Knot Floer homology (Ozsváth-Szabó, Rasmussen, 2003)

- $K \subset S^3$ a knot → bigraded abelian group $\widehat{HFK}_{i,j}(K)$.
- $\sum_{i,j}(-1)^i t^j rank\left(\widehat{HFK}_{i,j}(K)\right) = \Delta_K(t)$
- Homology of a chain complex $(\widehat{CFK}_{i,j}(K), \partial: \widehat{CFK}_{i,j}(K) \to \widehat{CFK}_{i-1,j}(K))$.
- Recall: $width(\Delta_K(t))/2 \leq g(K)$.
- **Theorem.** (Ozsváth-Szabó) $\frac{width\left(\widehat{HFK}_{i,j}(K)\right)}{2} := \max\{j \mid \widehat{HFK}_{*,j} \neq 0\} = g(K)$.

- Also works for null-homologous knots in other 3-manifolds.
- There are also versions for links.
- There are other variants -- $HFK^-(K)$, $HFK^+(K)$, etc.
- There are extensions to *sutured manifolds* (Juhász, Alishahi-Eftekhary).

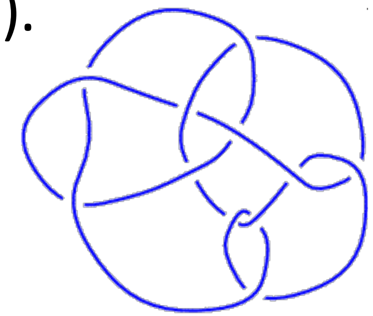


	-2	-1	0
1			Z
0		Z	
-1	Z		

$\Delta_K(t) = -t + 1 - t^{-1}$

$\Delta_K(t) = 1$

	-2	-1	0	1	2	3
2					Z	Z
1				Z^4	Z^4	
0			Z^7	Z^6		
-1		Z^4	Z^4			
-2	Z	Z				



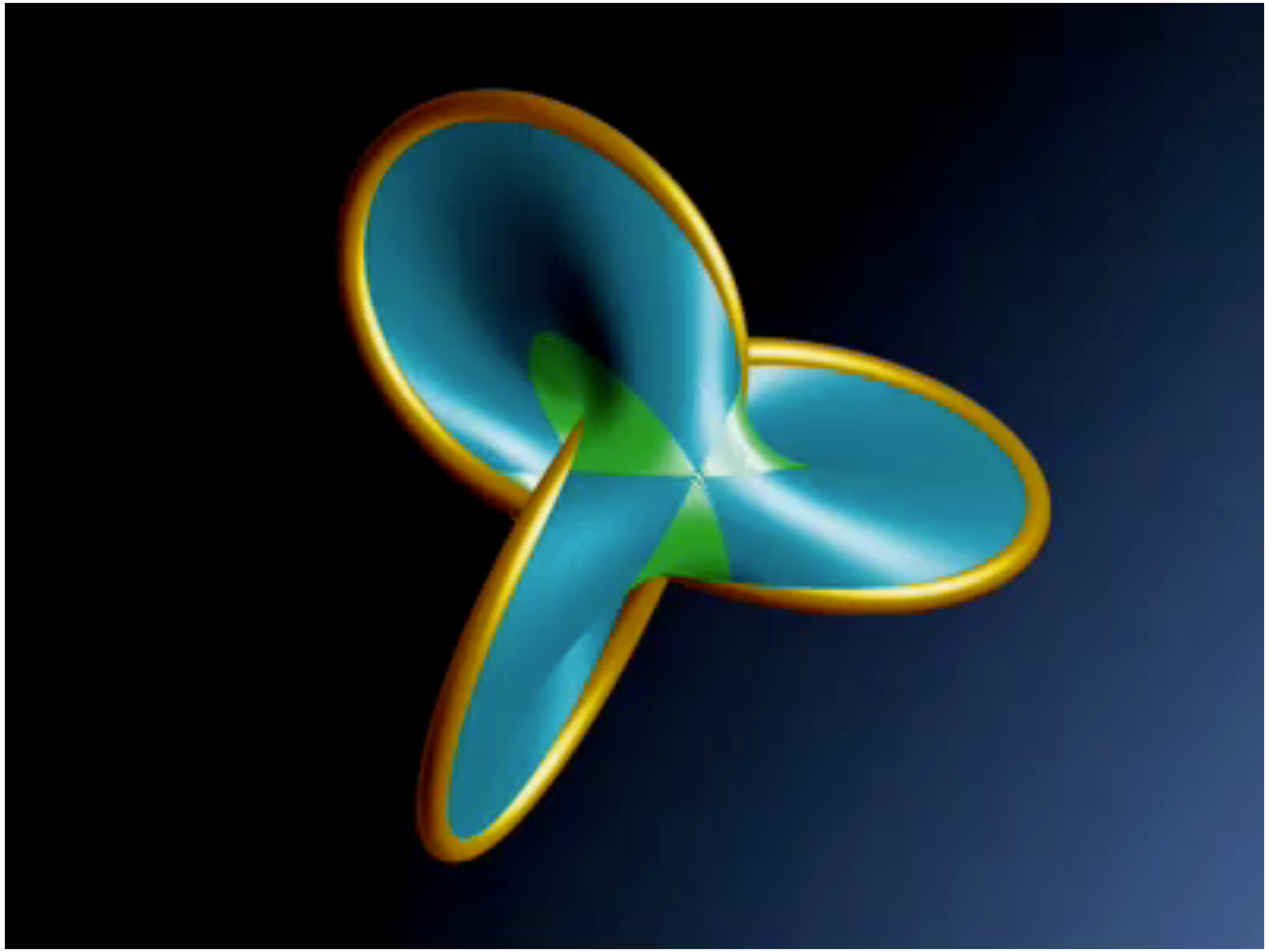
Kinoshita-Terasaka knot

Definition of knot Floer homology

- Via pseudo-holomorphic curves (solutions to particular nonlinear PDE's) in a high-dimensional auxiliary space.
- There are now combinatorial definitions (Manolescu-Ozsváth-Sarkar 2006, Manolescu-Ozsváth-Szabó-Thurston 2006, …)
- No classical definition is known. (In particular, not the singular homology of any naturally associated space.)

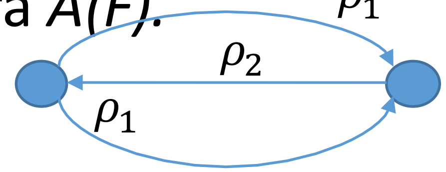
Fibered knots

- A knot is *fibered* if it has an S^1 family of Seifert surfaces.
- **Lemma.** If K is fibered, fiber F, monodromy ϕ then $\Delta_K(t)$ is the characteristic polynomial of $\phi_*: H_1(F) \to H_1(F)$.
- **Corollary.** If K is fibered then $\Delta_K(t)$ is monic and $width(\Delta_K(t)) = 2g(K)$.
- **Theorem.** (Ozsváth-Szabó, Ghiggini, Ni) $\widehat{HFK}(K)$ is monic if and only if K is fibered.
- (Monic means $\bigoplus_i \widehat{HFK}_{i,g(K)}(K) \cong \boldsymbol{Z}$.)



http://www.josleys.com/show_gallery.php?galid=303

Lifting the characteristic polynomial formula

- Bordered Floer homology (L-Ozsváth-Thurston):
  - Surface $F \rightsquigarrow$ dg algebra $A(F)$.
    - $A(S^2) = \boldsymbol{F}_2$. $A(T^2) =$ 

$/\rho_2\rho_1 = \rho_3\rho_2 = 0$ $H_*(A(\Sigma_2))$ 164 diml.

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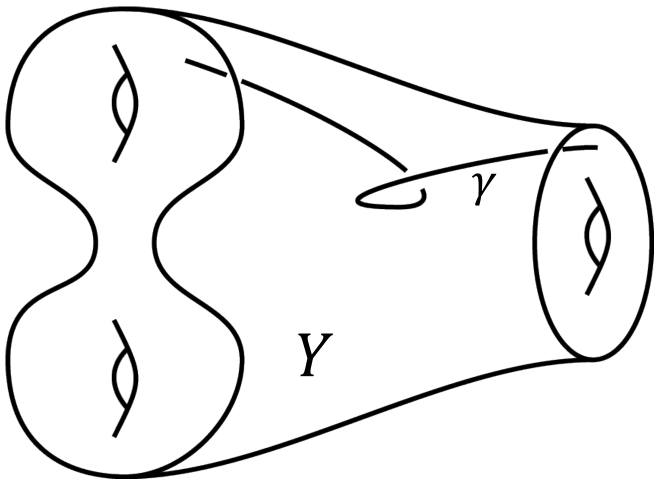
    - $A(S^2) = \boldsymbol{F}_2$. $A(T^2)$ 10-dimensional, $H_*(A(\Sigma_2))$ 164 dimensional.
  - Cobordism (Y, γ): $(F_1, pt) \to (F_2, pt)$ ⇝ dg bimodule $\widehat{CFDA}(Y, \gamma)$.



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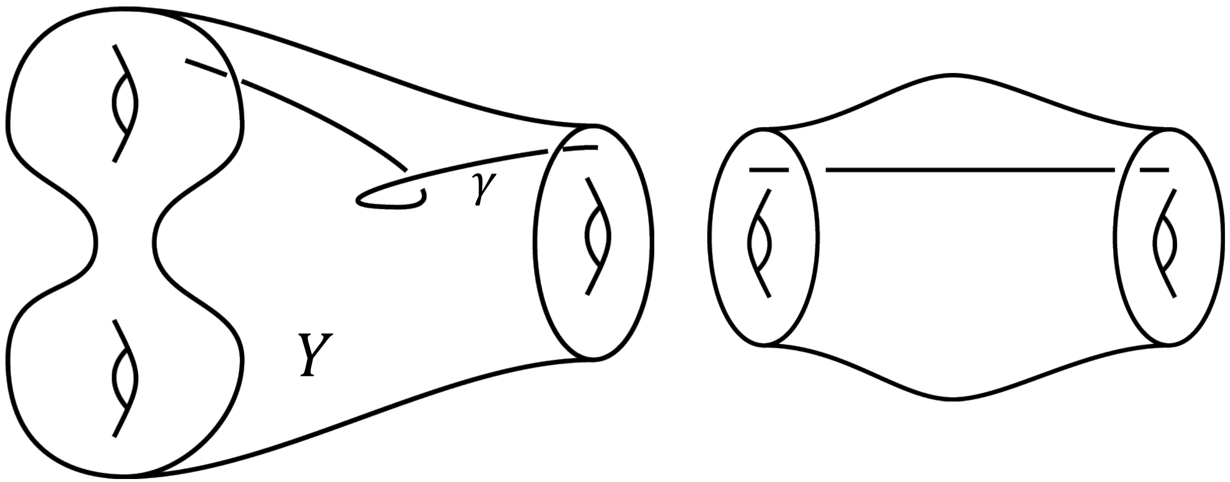
  - Composition ⇝ derived tensor product.



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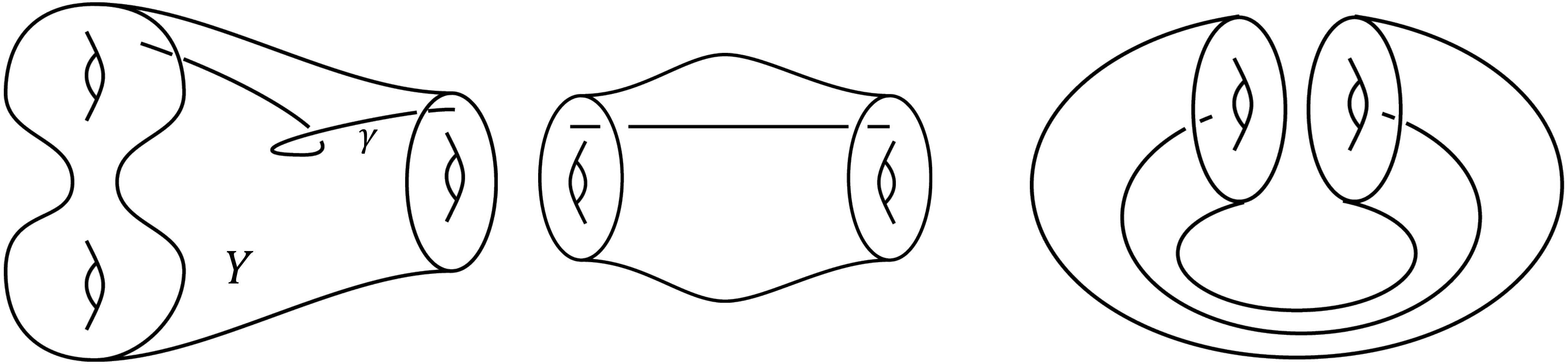
  - $F_1 = -F_2$ then self gluing (Y°, K) has $\widehat{HFK}\ (Y^\circ, K) = HH_*\left(\widehat{CFDA}(Y, \gamma)\right)$.



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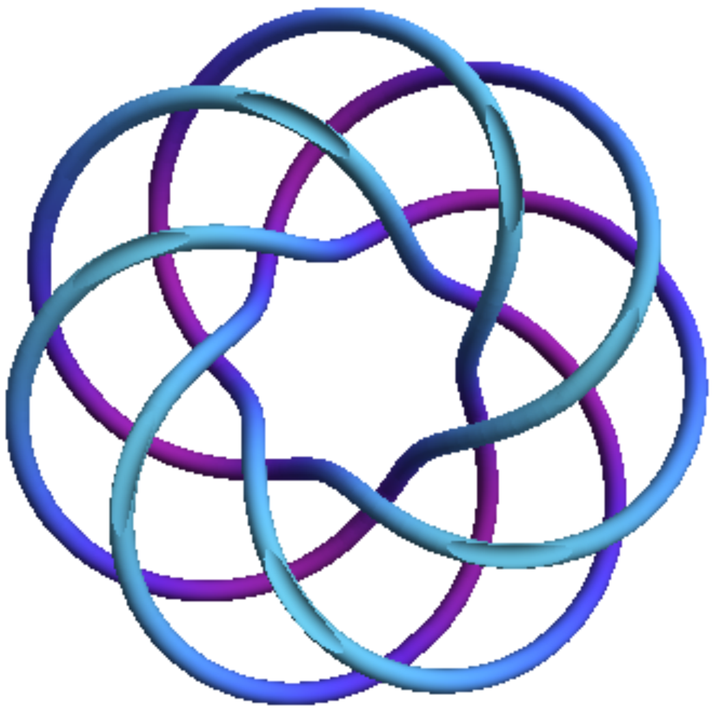
  - $F_1 = -F_2$ then self gluing (Y°, K) has $\widehat{HFK}\,(Y^\circ, K) = HH_*\left(\widehat{CFDA}(Y,\gamma)\right)$.
- Recall: for K fibered, $\Delta_K(t)$ is the characteristic polynomial of $\phi_*: H_1(F) \to H_1(F)$.
- Y_ϕ =mapping cylinder of ϕ then

L-Ozsváth-Thurston, Petkova, Hom-Lidman-Watson:

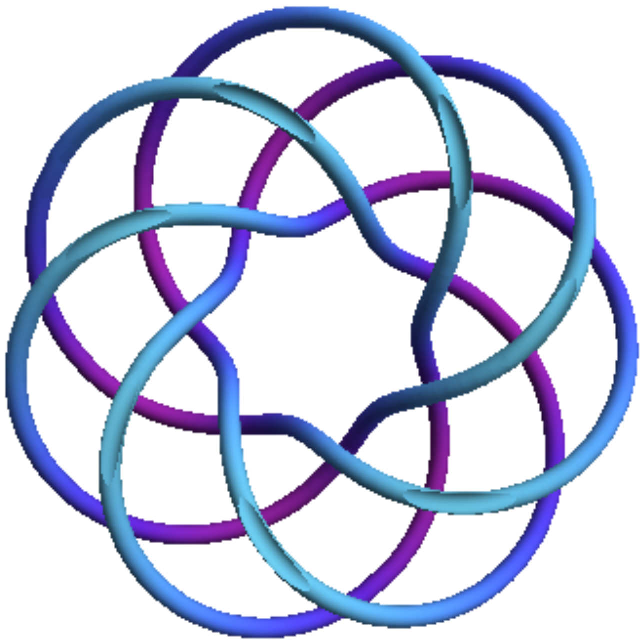
$$\begin{array}{ccc} \widehat{CFDA}(Y_\phi,\gamma) & \xrightarrow{HH_*} & \widehat{HFK}(S^3,K) \\ \big\downarrow K_0 & & \big\downarrow \chi \\ \phi_*: \Lambda^* H_1(F) \to \Lambda^* H_1(F) & \xrightarrow{\text{Trace}} & \Delta_K(t) \end{array}$$

Symmetries

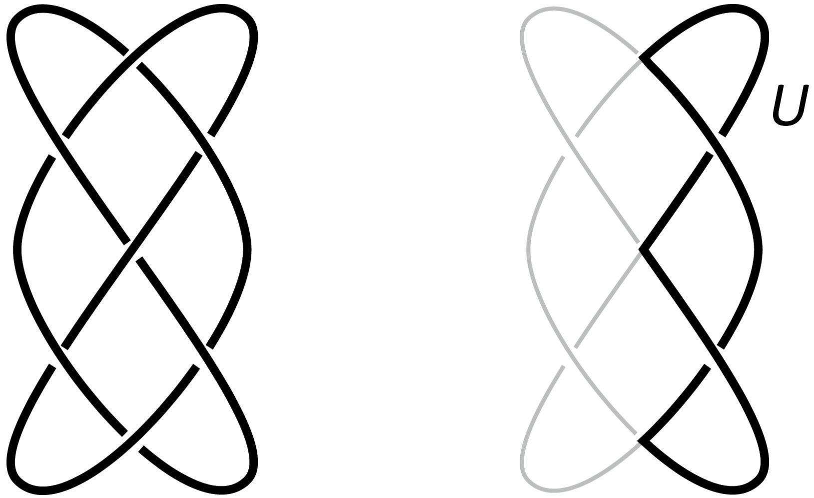


Periodic and Freely Periodic Knots

- K is *n-periodic* if it is preserved by rotation by $2\pi/n$ around an axis.
- K is *freely periodic of period (p,q)* if it is preserved by $(z,w) \mapsto (e^{\frac{2\pi i}{q}} z, e^{\frac{2\pi i p}{q}} w)$ *on* $S^3 = \{(z,w) \in \boldsymbol{C}^2 | |z|^2 + |w|^2 = 1\}$



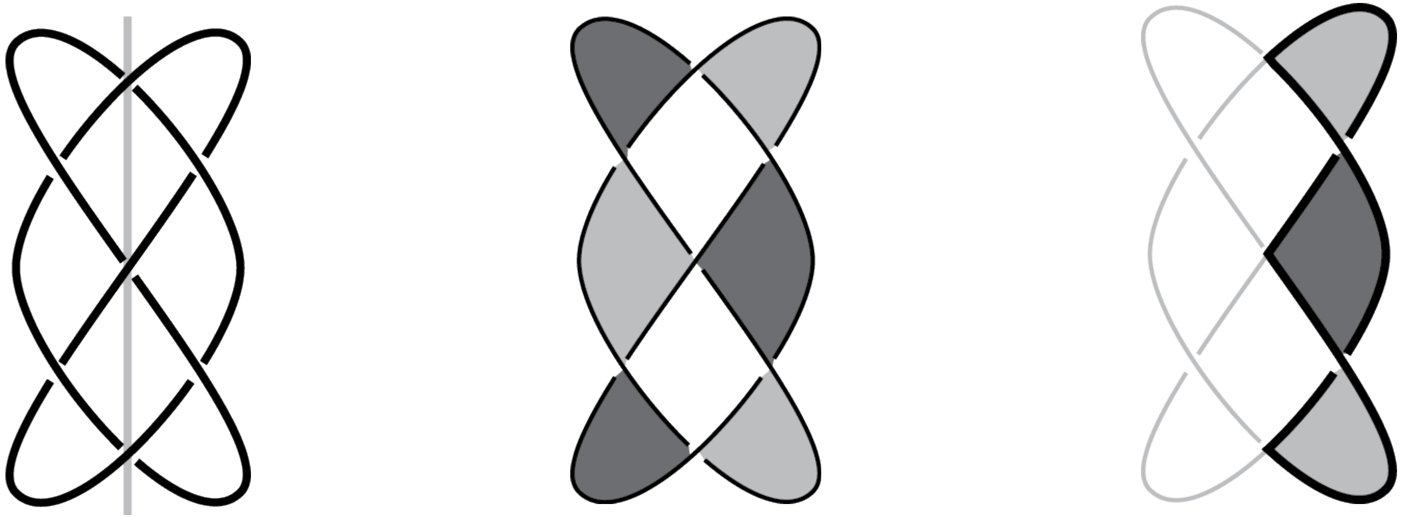
T(4,7) is 7-periodic, and freely (4,7) periodic.



7_4 is 2-periodic with *quotient* the unknot U.

Edmonds's and Murasugi's Conditions

- **Theorem**. (Edmonds, 1984) If *K* is *n*-periodic then there is a minimal genus Seifert surface for *K* preserved by ***Z/nZ.***
- (Proof uses minimal surfaces.)
- **Corollary**. If $\overline{K}$ is the quotient knot then $g(K) \geq ng(\overline{K}) + (n-1)(\lambda - 1)/2$ (where $\lambda =$ linking number with axis).



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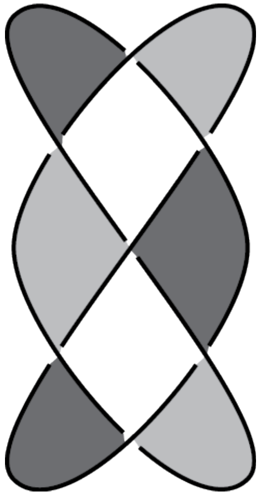
- **Corollary**. If $\overline{K}$ is the quotient knot then $g(K) \geq ng(\overline{K}) + (n-1)(\lambda - 1)/2$ (where $\lambda =$ linking number with axis).
- **Corollary**. Any nontrivial knot has finitely many periods.
- **Theorem**. (Mursagui, 1971) If *K* is p^r-periodic then $\Delta_K(t) \equiv \pm\, t^i \Delta_{\overline{K}}(t)^n \left(\frac{1-t^\lambda}{1-t}\right)^{n-1} (mod\, p)$.



A theorem of Hendricks

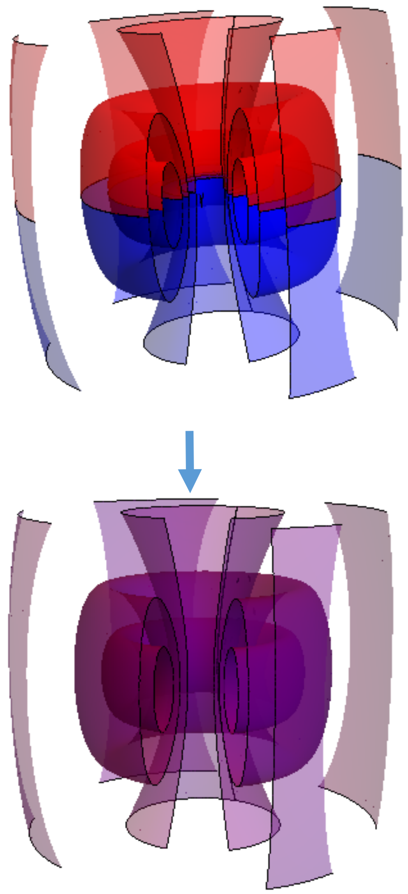
- **Theorem**. [(Hendricks, 2012) + ϵ(Hendricks-L-Sarkar 2015)] If K is 2-periodic then there is a spectral sequence

$$\widehat{HFL}(K \cup A) \otimes (\boldsymbol{F_2} \oplus \boldsymbol{F}_2) \otimes \boldsymbol{F}_2[\theta, \theta^{-1}] \Rightarrow \widehat{HFL}\left(\overline{K} \cup \overline{A}\right) \otimes \boldsymbol{F_2}[\theta, \theta^{-1}].$$

- Proof uses Seidel-I. Smith's quantum (P.A.) Smith Theory.
- Implies (lifts) 2-periodic cases of Edmonds's Corollary and Murasugi's Theorem.
- Has other implications as well.

Branched double covers

- The *double cover of* S^3 *branched along* K is
$$\pi: \left(\Sigma(S^3, K), \widetilde{K}\right) \to (S^3, K)$$
  - $\pi|_{\Sigma(S^3,K)\setminus\widetilde{K}}: \left(\Sigma(S^3,K) \setminus \widetilde{K}\right) \to (S^3 \setminus K)$ is a 2-fold cover.
  - In the normal planes to K, π *is* $z \mapsto z^2$.
- Special cases:
  - $\Sigma(S^3, U) = (S^3, U)$.
  - If (K, A) is 2-periodic then $(S^3 \setminus K, A) = \Sigma\left(S^3 \setminus \overline{K}, \overline{A}\right)$.
- Smith Conjecture (proved in 1980's) states that if $K \neq U$ then $\Sigma(S^3, K) \neq S^3$.
- Can also talk about $\Sigma(Y^3, K)$, but not always unique.



Knot Floer homology in branched double covers

- **Classical?:** For $\left(\Sigma(K), \widetilde{K}\right) \to (S^3, K)$, $\Delta_{\widetilde{K}}(t) \equiv \Delta_K(t)\ (mod\ 2)$**.**
- **Theorem**. (Hendricks, 2011) There is a spectral sequence
  $$\widehat{HFK}\left(\Sigma(S^3, K), \widetilde{K}\right) \otimes \boldsymbol{F}_2[\theta, \theta^{-1}] \Rightarrow \widehat{HFK}(S^3, K) \otimes \boldsymbol{F}_2[\theta, \theta^{-1}].$$
- **Theorem**. (L-Treumann, 2012) If $H_1(Y) = 0$ and $K \subset Y$ has $g(K) \leq 2$ then there is a spectral sequence
  $$\widehat{HFK}\left(\Sigma(Y, K), \widetilde{K}\right) \otimes \boldsymbol{F}_2[\theta, \theta^{-1}] \Rightarrow \widehat{HFK}(Y, K) \otimes \boldsymbol{F}_2[\theta, \theta^{-1}].$$

Proof sketch via bordered Floer.

- Cut *Y* along Seifert surface *F* for *K* to get cobordism *Z* from *F* to *F*.
- (S^3, K) is self-gluing of *Z*. $(\Sigma(K), \widetilde{K})$ is self-gluing of $Z \cup_F Z$.
- So, want:

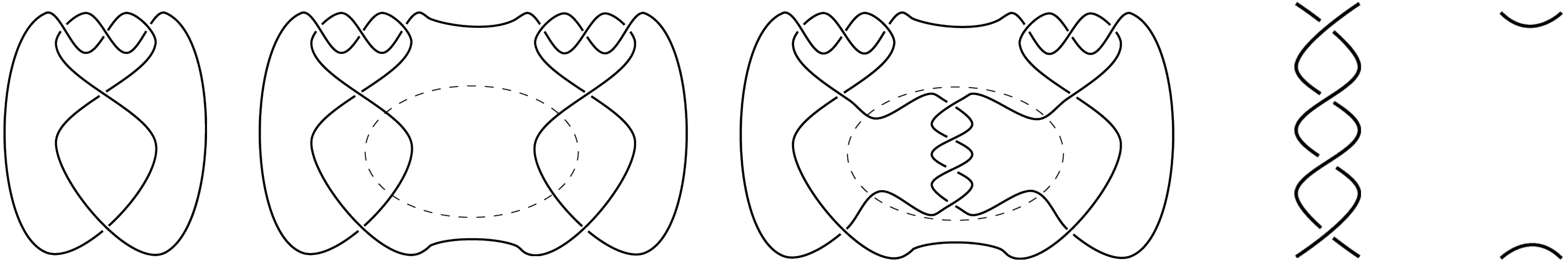
$$\widehat{HFK}\big(\Sigma(K), \widetilde{K}\big) = HH_*\big(CFDA(Z) \otimes^L_{A(F)} CFDA(Z)\big)$$
$$\Rightarrow HH_*\big(CFDA(Z)\big) = \widehat{HFK}(S^3, K)$$

- (Suppressed copies of $\boldsymbol{F}_2[\theta, \theta^{-1}]$.)
- Such a spectral sequence exists whenever $A(F)$ satisfies certain algebraic properties.

Honest covers

- If $\tilde{\tilde{Y}} \to Y$ is a $\boldsymbol{Z}$ cover then there is an induced $\boldsymbol{Z}/2\boldsymbol{Z}$ cover
$$(\tilde{\tilde{Y}}/2\boldsymbol{Z}) = \tilde{Y} \to Y$$
- **Theorem**. (L-Treumann, 2012) If $\tilde{Y} \to Y$ is a $\boldsymbol{Z}/2\boldsymbol{Z}$ cover induced by a $\boldsymbol{Z}$ cover then $\widehat{HF}(\tilde{Y}) \otimes (\boldsymbol{F}_2 \oplus \boldsymbol{F}_2) \otimes \boldsymbol{F}_2[\theta, \theta^{-1}] \Rightarrow \widehat{HF}(Y) \otimes \boldsymbol{F}_2[\theta, \theta^{-1}]$.
- **Theorem**. (Lidman-Manolescu, 2016) If $\tilde{Y} \to Y$ is a $\boldsymbol{Z}/2\boldsymbol{Z}$ cover and $H^1(Y) = 0$ then for each $s \in spin^c(Y)$,
$$\widehat{HF}(\tilde{Y}, \pi^* s) \otimes \boldsymbol{F}_2[\theta, \theta^{-1}] \Rightarrow \widehat{HF}(Y, s) \otimes \boldsymbol{F}_2[\theta, \theta^{-1}].$$
- **Corollary**. $rank\left(\widehat{HF}(\tilde{Y}, \pi^* s)\right) \geq rank\left(\widehat{HF}(Y, s)\right)$.
- Lidman-Manolescu is uses a variant of Seiberg-Witten Floer homology. Main work is identifying this variant with more common (and computable) ones.
- Some applications of Lidman-Manolescu's result later in the talk.

Symmetric Unions



J $\quad$ $J\#m(J)$ $\quad$ $K_n(J)$ $\quad$ T_n $\quad$ T_∞

- **Theorem**. (Kinoshita-Terasaka, 1957) If n is even then $\Delta_{K_n(J)}(t) = (\Delta_J(t))^2$.
- **Theorem**. (Allison Moore, 2015) If replacing T_n with T_∞ gives a 2-component unlink then $\widehat{HFK}\big(K_n(J)\big) \cong \widehat{HFK}(J) \otimes \widehat{HFK}(m(J))$.
- **Corollary**. $g\big(K_n(J)\big) = 2g(J)$ and $K_n(J)$ is fibered iff J is.
- **Corollary**. New examples satisfying *cosmetic crossing conjecture*: any non-nugatory crossing change changes isotopy class of K.

More Alexander polynomial formulas

- Symmetric knots (K-T): $\Delta_{K_n(J)}(t) = (\Delta_J(t))^2$.
- If $\Sigma^n(K)$ is n-fold cyclic branched cover of S^3 then

  $\left|H_1\left(\Sigma^n(K)\right)\right| = \prod_{j=0}^{n-1} \Delta_K(e^{2\pi i j/n})$. e.g., $\left|H_1\left(\Sigma(K)\right)\right| = \Delta_K(1)\Delta_K(-1)$.
- If $\left(\Sigma^n(K), \widetilde{K}\right)$ *n*-fold cyclic branched cover of (S^3, K) then

  $\Delta_{\widetilde{K}}(t^n) = \prod_{j=0}^{n-1} \Delta_K\left(e^{\frac{2\pi i j}{n}} t\right)$. e.g., if $n = 2$, $\Delta_{\widetilde{K}}(t^2) = \Delta_K(t)\Delta_K(-t)$.
- (Murasugi) If $\widetilde{K}$ is n-periodic, axis $\tilde{A}$, quotient (K, A), $\lambda = lk(K, A)$ then

$$\frac{1 - t^\lambda}{1 - t} \Delta_{\widetilde{K}}(t) = \prod_{j=0}^{n-1} \Delta_{K\cup A}\left(t, e^{\frac{2\pi i j}{n}}\right).$$

- (Hartley, 1981) Similar result for freely periodic knots.
- **Question**. Can any of these be lifted to Floer homology? How?

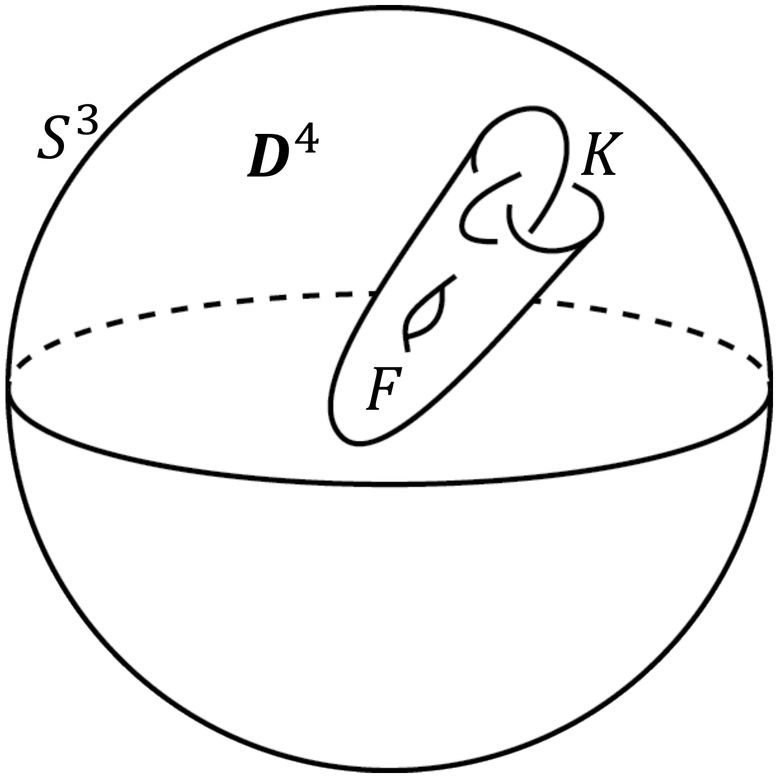
More open questions about Floer homology, Seifert surfaces, and symmetries

- There are nice combinatorial definitions of $\widehat{HFK}(S^3, K)$. Can one give combinatorial proofs of key properties, like detecting $g(K)$ or fiberedness?
- Edmonds's condition says a minimal genus Seifert surface is $\boldsymbol{Z}/n\boldsymbol{Z}$ equivariant. Can one prove this with Floer homology? (Recall that Hendricks proved the main numerical corollary.)
- For any knot K in a homology sphere Y, is there a spectral sequence $\widehat{HF}\big(\Sigma(K)\big) \Rightarrow \widehat{HF}(Y)$? (cf. Smith conjecture.)

Concordance

The concordance group and slice genus

- **Definition**. The *smooth slice genus* of K is
  $g_4(K) = \min\{g(F) | F \subset \boldsymbol{D}^4, \partial F = K \subset S^3 = \partial \boldsymbol{D}^4, F\ smooth\}$.



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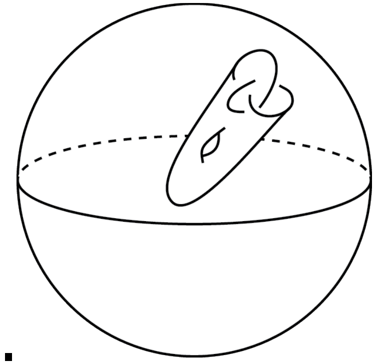

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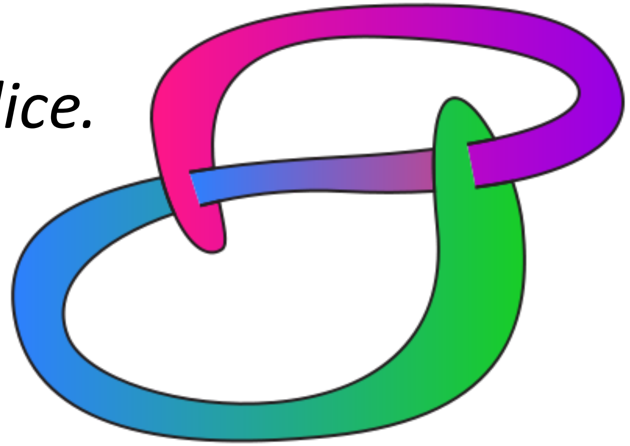
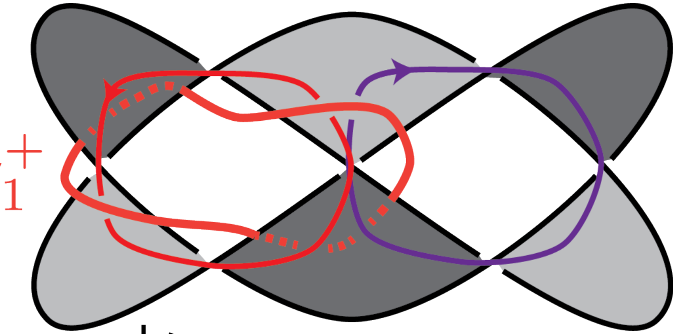
  $g_4(K) = \min\{g(F) | F \subset \boldsymbol{D}^4, \partial F = K \subset S^3 = \partial \boldsymbol{D}^4, F\ smooth\}$.
- K is *slice* if $g_4(K) = 0$.
- **Definition.** The smooth concordance group is
  $\mathcal{C} = (\{knots\}/\{slice\ knots\}, \#)$.
- Replace "smooth" by "topologically locally flat" gives *topological slice genus* $g_4^{top}(K)$, *topologically slice knots, topological concordance group* $\mathcal{C}_{\mathcal{T}op}$.
- $g_4^{top}(K) \leq g_4(K)$, *so smoothly slice implies topologically slice.*
- $0 \to \mathcal{C}_{\mathcal{TS}} \to \mathcal{C} \to \mathcal{C}_{\mathcal{T}op} \to 0$.



David Eppstein
"Square Ribbon Knot"
Wikipedia

Two classical restrictions



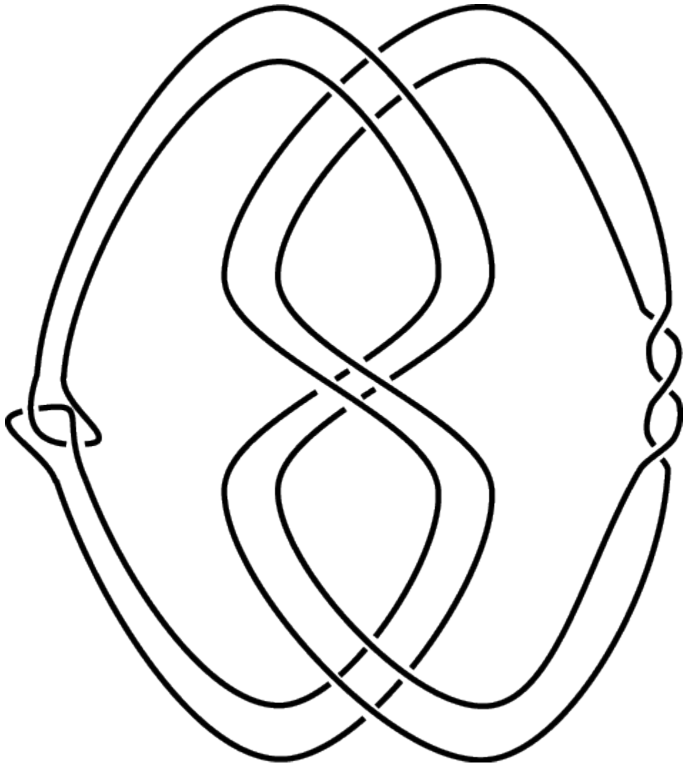
- Recall: the Seifert matrix $A = (a_{i,j})$ with $a_{i,j} = lk(\gamma_i, \gamma_j^+)$.
- **Definition**. The *signature* $\sigma(K) \in \boldsymbol{Z}$ is the signature of $A + A^T$.
- **Theorem**. (Kauffman-Taylor, Murasugi, Tristram, 1960's) The signature is a homomorphism $\mathcal{C}_{\mathcal{T}op} \to \boldsymbol{Z}$, and $2g_4^{top}(K) \geq \sigma(K)$.
- **Theorem**. (Fox-Milnor, 1966) If K is topologically slice then there is a polynomial $f(t)$ so that $\Delta_K(t) = f(t)f(t^{-1})$.
- **Theorem**. (Levine, 1969) There is a surjection $\mathcal{C}_{\mathcal{T}op} \to \boldsymbol{Z}^\infty \oplus (\boldsymbol{Z}/2\boldsymbol{Z})^\infty \oplus (\boldsymbol{Z}/4\boldsymbol{Z})^\infty$.
- **Question**. Is there any torsion other than 2-torsion in $\mathcal{C}$ or $\mathcal{C}_{\mathcal{T}op}$?

Some modern concordance homomorphisms

- Turning to smooth concordance:
- **Theorem**. (Endo, 1995) There is a $\boldsymbol{Z}^\infty$ subgroup of $\mathcal{C}_{\mathcal{TS}} = \ker(\mathcal{C} \to \mathcal{C}_{\mathcal{Top}})$.
- Does it split? Surjections *onto* $\boldsymbol{Z}$? Bounds on g_4?
  - Ozsváth-Szabó, 2003. τ: $\mathcal{C} \to \boldsymbol{Z}$, via Heegaard Floer homology. Bounds g_4.
  - Rasmussen, 2004. s: $\mathcal{C} \to \boldsymbol{Z}$, via Khovanov homology. Bounds g_4.
  - Manolescu-Owens, 2005. δ: $\mathcal{C} \to \boldsymbol{Z}$, via Heegaard Floer homology.
  - Livingston, 2006. τ, s, δ give a surjection $\mathcal{C}_{\mathcal{TS}} \to \boldsymbol{Z}^3$.
  - Hom, 2013. A surjection $\mathcal{C}_{\mathcal{TS}} \to \boldsymbol{Z}^\infty$. (Not constructive.)
  - Ozsváth-Stipsicz-Szabó, 2014. Another surjection Υ: $\mathcal{C}_{\mathcal{TS}} \to \boldsymbol{Z}^\infty$. (Constructive.)
  - Hendricks-L-Sarkar, 2015. Another homomorphism q_τ: $\mathcal{C} \to \boldsymbol{Z}$, via equivariant Floer homology.
  - Many other beautiful results by many other researchers.

An explicit example in $\mathcal{C}_{\mathcal{TS}}$

- Recall: $0 \to \mathcal{C}_{\mathcal{TS}} \to \mathcal{C} \to \mathcal{C}_{\mathcal{T}op} \to 0$.
- **Theorem**. (Freedman-Quinn ~1989) If $\Delta_K(t) = 1$ then K is topologically slice.
- **Exercise**. The *(positive) untwisted Whitehead double* $D_+(K)$ of any knot K has $\Delta_{D_+(K')}(t) = 1$.
- **Theorem**. (Hedden 2006) If $\tau(K) > 0$ then $\tau\big(D_+(K)\big) = 1$.
- **Corollary**. $\mathcal{C}_{\mathcal{TS}}$ is nontrivial.
- (This is not the first proof, which is attributed to Casson (unpublished). Gompf gave first examples of non-slice Whitehead doubles.)



$\tau\left(D_+\big(T(2,3)\big)\right) = 1.$

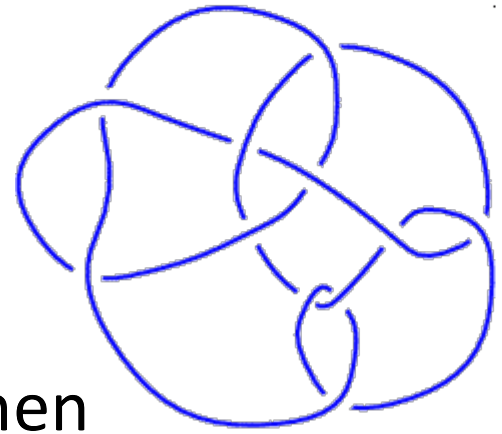
From $\mathcal{C}_{\mathcal{TS}}$ to exotic $\boldsymbol{R}^4$'s

- **Theorem**. (Classical, Moise-Munkres ~1960, Stallings 1961) There is a unique smooth structure on $\boldsymbol{R}^n$ is $n \leq 2, n = 3, n > 4$.
- **Theorem**. (Taubes, 1987) There are uncountably many smooth structures on $\boldsymbol{R}^4$.
- **Theorem**. (Freedman-Quinn) Any non-compact 4-manifold has a smooth structure.

Here's an exotic $\boldsymbol{R}^4$ (Gompf-Stipsicz, Exercise 9.4.23):

- Take a knot K in $\mathcal{C}_{\mathcal{TS}}$. Let X_K be result of attaching a 2-handle to $\boldsymbol{D}^4$ along K. There is a topologically flat embedding $X_K \hookrightarrow \boldsymbol{R}^4$.
- Freedman-Quinn implies $\boldsymbol{R}^4 \setminus X_K$ can be smoothed. Glue X_K to this smoothing of $\boldsymbol{R}^4 \setminus X_K$ to get a smooth $\boldsymbol{R}^4$. Must be exotic, because K is smoothly slice in it.

Lifting the Fox-Milnor condition



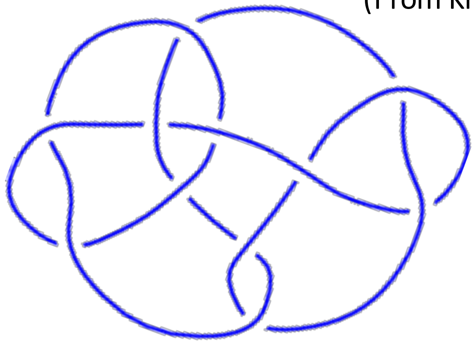
K11n42, mirror of KT, from Knot Atlas

- Recall: the Fox-Milnor condition: if *K* is topologically slice then $\Delta_K(t) = P(t)P(t^{-1})$.
- Question. Is there a lift of this result to $\widehat{HFK}$?
  - Presumably such a result would be about *smoothly* slice knots.
  - The Kinoshita-Terasaka knot is topologically slice (since $\Delta_K(t) = 1$) but $\widehat{HFK}(K)$ has Poincaré polynomial $(q^{-2} + q^{-1})t^{-2} + 4(q^{-1} + t)t^{-1} + 7 + 6q + 4(q + q^2)t + (q^2 + q^3)t^2$, and total rank $2 + 8 + 7 + 6 + 8 + 2 = 33$ (not square!).
  - Maybe some sort of spectral sequence?
  - Maybe only four *doubly slice* knots (knots with an invertible concordance to *U*)?

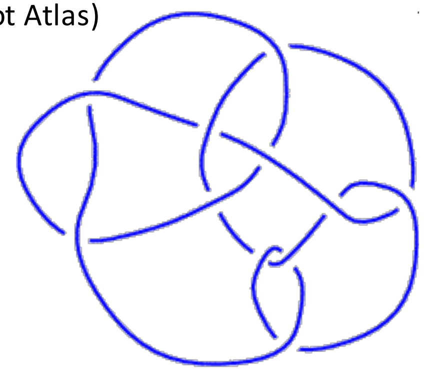
Some more open concordance questions

- Is there a surjection $\mathcal{C}_{\mathcal{TS}} \to (\boldsymbol{Z}/2\boldsymbol{Z})^{\infty}$? Any other torsion?
- **Conjecture**. (Kirby 1.38) $D_+(K)$ is smoothly slice if and only if K is.
- **Sub-conjecture**. The positive Whitehead double of the negative trefoil is not smoothly slice.
- What is the smooth slice genus of the Conway knot? (Its mutant, the Kinoshita-Terasaka knot, is slice. $\Delta_{Conway}(t) = \Delta_{KT}(t) = 1$.)

(From Knot Atlas)

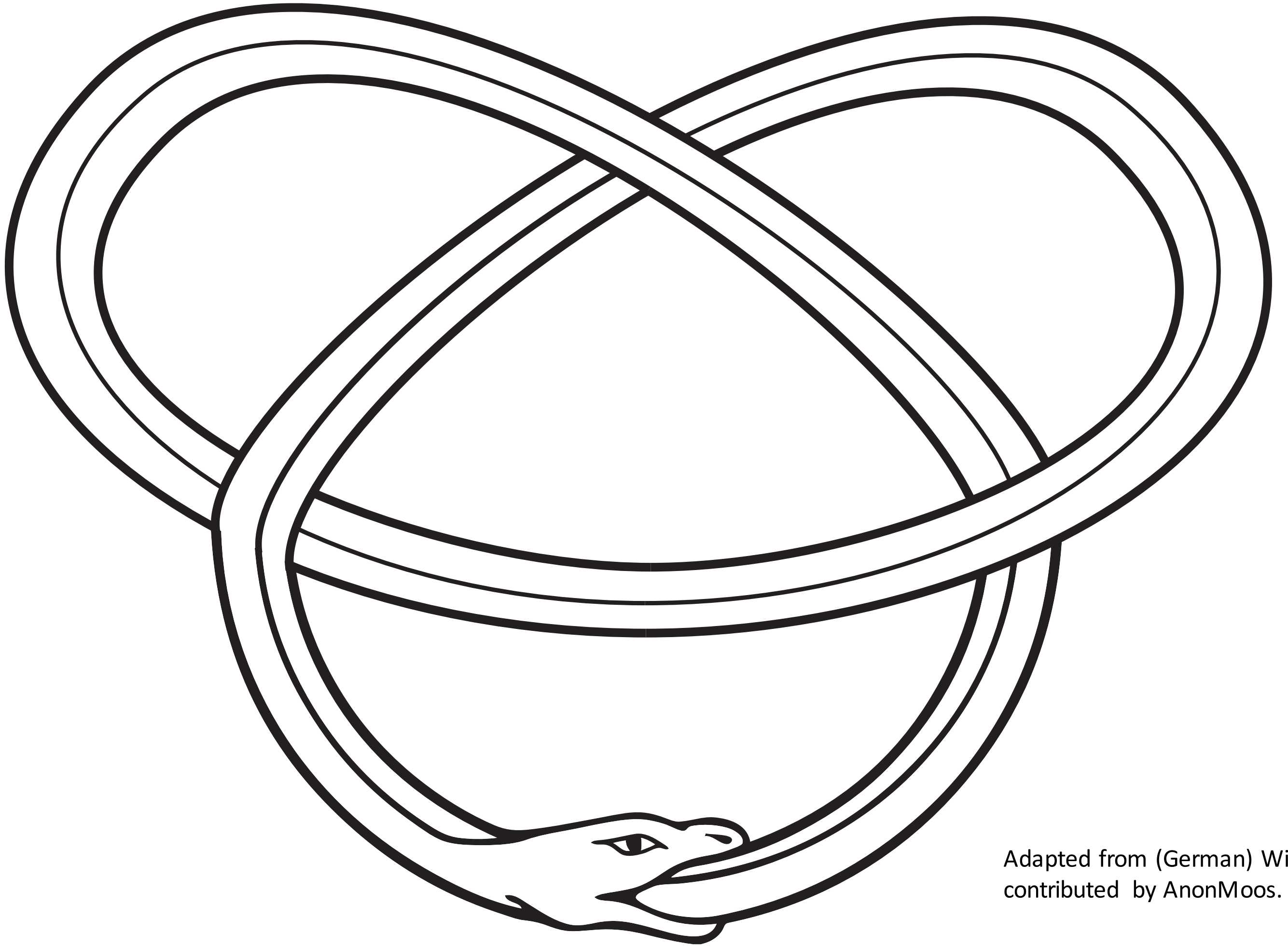


Conway knot



Kinoshita-Terasaka knot

From Interstellar movie poster
(Paramount Pictures)

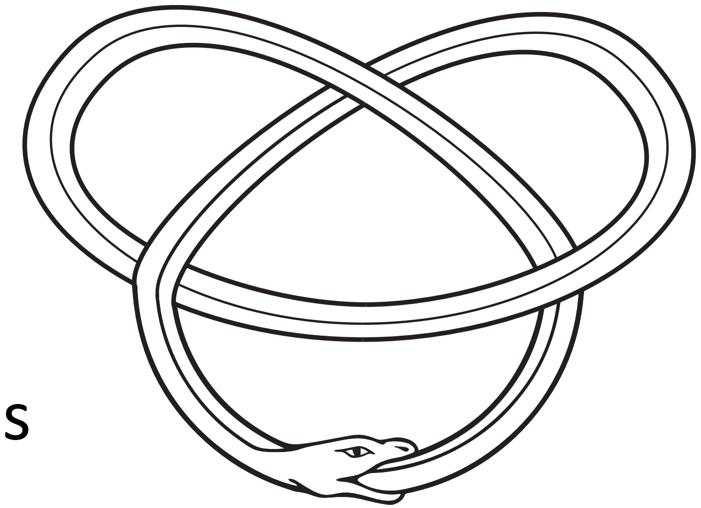


Adapted from (German) Wikipedia entry,
contributed by AnonMoos.

Dehn surgery

is Ouroboros holes.

Definition of Dehn surgery



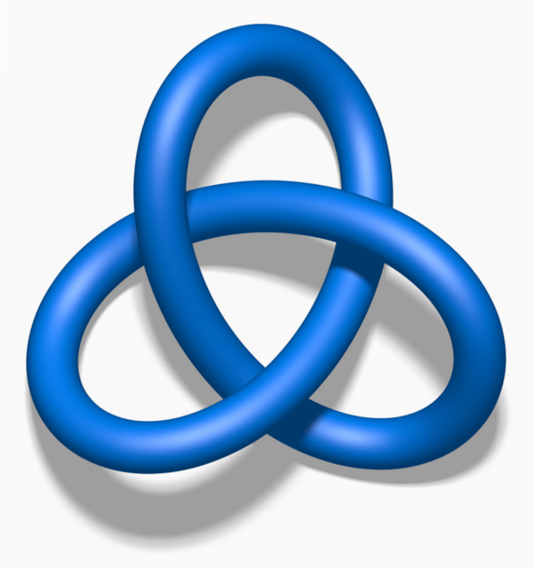
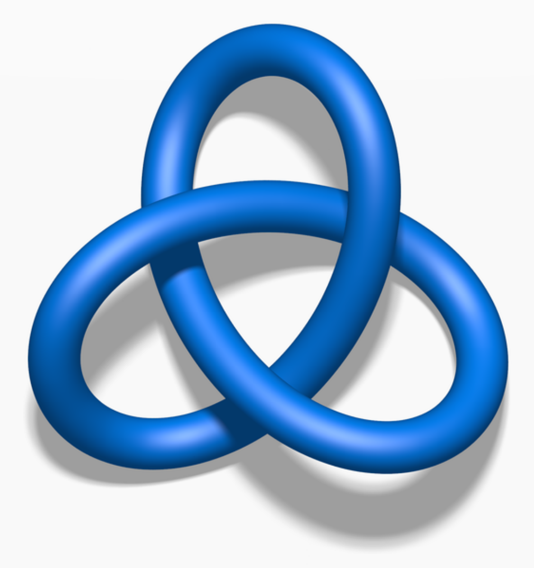
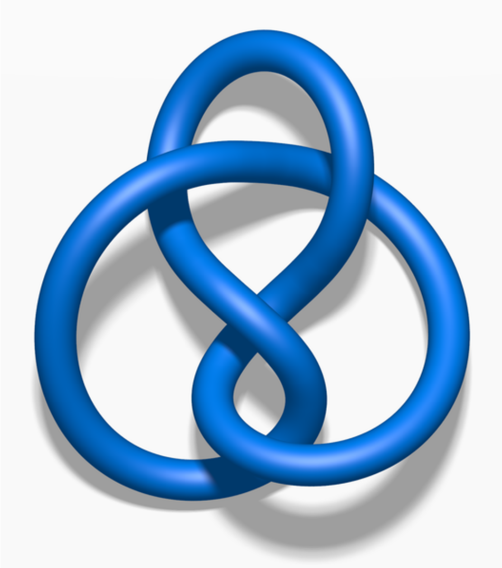
- **Definition**. *Dehn surgery* on a link $L = K_1 \cup \cdots \cup K_n$ is $S^3 \setminus nbd(K_1 \cup \cdots K_n) \cup_\partial (\boldsymbol{D}^2 \times S^1 \cup \cdots \cup \boldsymbol{D}^2 \times S^1)$.
- $\partial\big(S^3 \setminus nbd(K_1 \cup \cdots K_n)\big) = T^2 \coprod \cdots \coprod T^2$. Surgery is determined by images of the n circles $\partial \boldsymbol{D}^2 \times \{pt\}$ which bound disks $\boldsymbol{D}^2$.
- Each circle is $p \cdot (meridian) + q \cdot (longitude)$, so have
$$S^3_{\frac{p_1}{q_1}, \cdots, \frac{p_n}{q_n}}(L).$$
- **Theorem**. (Lickorish, 1962) Every closed, orientable 3-manifold is $S^3_{\frac{p_1}{q_1}, \cdots, \frac{p_n}{q_n}}(L)$ for some $\frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n}$ and L.

Some questions and some answers

- **Question**. Which manifolds arise as surgery on a knot?
  - There are some obvious restrictions on H_1, π_1.
  - There are non-obvious restrictions on manifolds with nontrivial first homology (Boyer-Lines 1990).
  - **Theorem**. (Hom-Karakurt-Lidman, 2014) There are non-obvious restrictions on manifolds with $H_1 = 0$. e.g., $\Sigma(p, 2p-1, 2p+1) = \{ (z_1, z_2, z_3) \in S^5 \subset \boldsymbol{C}^3 \mid z_1^p + z_2^{2p-1} + z_3^{2p+1} = 0 \}$, $p \geq 8$ does not.
- **Question**. Are there non-obvious restrictions on which manifolds arise as surgery on an *n*-component link (*n>1* fixed)?

More questions and answers

- **Question**. If $S^3_{\frac{p}{q}}(K) \cong S^3_{\frac{p}{q}}(K')$ must $K = K'$?
  - **Theorem**. (Gabai, Kronheimer-Mrowka-Ozsváth-Szabó, 2003) Yes if *K*=*U*.
  - **Theorem**. (Ghiggini, Ozsváth-Szabó, 2006) Yes if $K = 3_1, m(3_1)$, or 4_1.

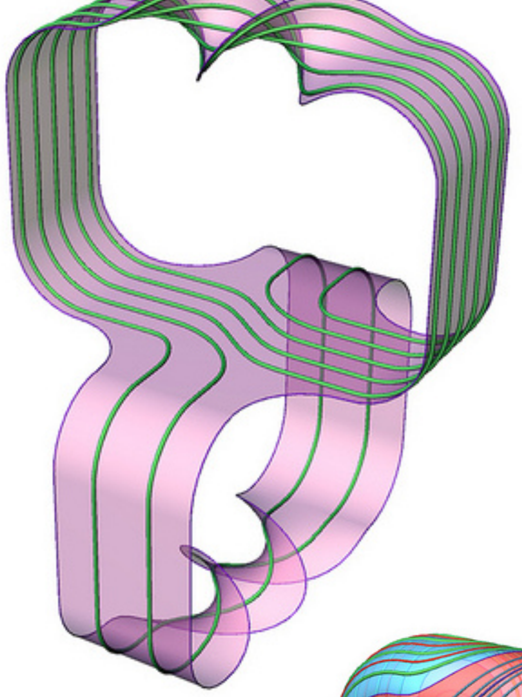
Knot Atlas / KnotPlot

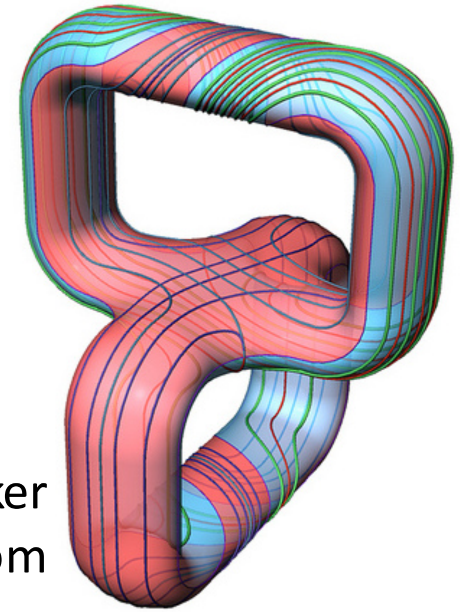
Surgery characterizations of knots

- **Question**. If $S^3_{\frac{p}{q}}(K) \cong S^3_{\frac{p}{q}}(K')$ must $K = K'$?
  - **Theorem**. (Gabai, Kronheimer-Mrowka-Ozsváth-Szabó, 2003) Yes if *K=U*.
  - **Theorem**. (Ghiggini, Ozsváth-Szabó, 2006) Yes if $K = 3_1$, $m(3_1)$, or 4_1.
  - "No" in general.
  - **Question**. What about other specific knots (e.g., $T_{3,4}$)?

Lens space surgeries

- **Question**. For which K is there p/q with $S^3_{p/q}(K)$ a lens space?
- **Berge Conjecture**. Such K is *doubly primitive*, i.e.,
  - Lies on a genus 2 Heegaard surface
  - Meets some meridian disk on each side once.
- **Theorem**. (Ni,2006): If $S^3_{p/q}(K)$ is a lens space then K is fibered.
- **Theorem**. (Greene, 2010): The lens spaces that arise from knot surgery are those predicted by Berge conjecture.





An application of equivariant Floer homology

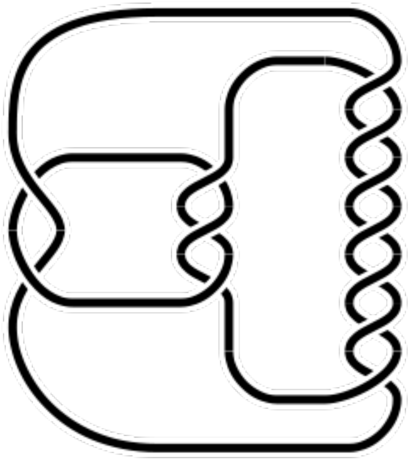
- **Question**. When does one surgery on a knot cover another?
- **Example**. $S^3_{\frac{6q+1}{q}}(3_1) = L(6q+1, 4q)$ (Moser, 1971). So $S^3_{\frac{6q+1}{q}}(3_1)$ is a $(6k+1)$-fold cover of $S^3_{\frac{6q'+1}{q'}}(3_1)$ for $q' = q + k(6k+1)q$.
- **Theorem**. (Lidman-Manolescu, 2016) If $\frac{p}{q} < 1$ and $\lceil q/p \rceil < \lfloor q'/p' \rfloor$ then $S^3_{\frac{p}{q}}(K)$ is not a regular r^n sheeted cover of $S^3_{\frac{p'}{q'}}(K)$ (r prime).
- (Proof uses equivariant Floer homology spectral sequence $\widehat{HF}(\tilde{Y}) \Rightarrow \widehat{HF}(Y)$ mentioned earlier.)

L-spaces and left orderability

L-spaces

- **Definition**. Y^3 is a *rational homology sphere* if $H_*(Y;\boldsymbol{Q}) \cong H_*(S^3,\boldsymbol{Q})$.
- **Theorem**. (Ozsváth-Szabó) If Y^3 is a rational homology sphere then
$$\chi\left(\widehat{HF}(Y)\right) = |H_1(Y)| = |spin^c(Y)|.$$
- **Definition**. A rational homology sphere Y^3 is an *L-space* if $\widehat{HF}(Y) \cong \boldsymbol{Z}^{|H_1(Y)|}$ (i.e., as small as possible).
- **Examples**.

Space	H_1	$\widehat{\boldsymbol{HF}}$
Poincaré homology sphere $S^3_1(3_1)$	0	$\boldsymbol{Z}$
Lens space $L(p,q) = S^3_{\frac{p}{q}}(U)$	$\boldsymbol{Z}/p\boldsymbol{Z}$	$\boldsymbol{Z}^{\boldsymbol{p}}$
(Any manifold with spherical geometry.)		
$\Sigma(K)$ for K alternating	$\lvert H_1\rvert = \det(K)$ $= \det(A + A^T)$	$\boldsymbol{Z}^{\det(K)}$
$S^3_p(P(-2,3,7))$ (Hyperbolic for p>19)	$\boldsymbol{Z}/p\boldsymbol{Z}$	$\boldsymbol{Z}/p\boldsymbol{Z}$

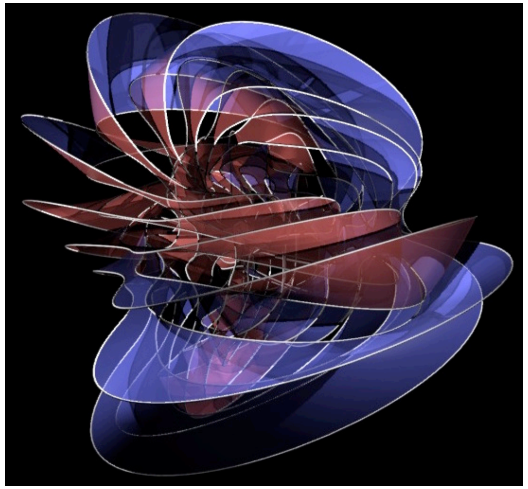

(From Wikipedia)

Some results for L-spaces

- Strangely, Floer theory is very useful for studying L-spaces. Some examples:
- **Theorem**. (Ozsváth-Szabó 2003) If $S^3_p(K)$ is an L-space then $\Delta_K(t) = (-1)^k + \sum_{j=1}^{k}(-1)^{k-j}(T^{n_j} + T^{-n_j})$ for an increasing sequence $0 < n_1 < \cdots < n_k$.
    - Strong restriction on what knots have lens space surgeries.
- **Theorem**. (Lidman-Moore 2015) If $\Sigma(K)$ is an L-space and $\det(K)$ is square-free then K satisfies the cosmetic crossing conjecture.
- **Theorem**. (Greene 2011) If K, K' are alternating and $\Sigma(K) \cong \Sigma(K')$ then K and K' are related by a sequence of mutations.
    - Partial converse to an old observation of Viro.
    - Uses (grading on) $\widehat{HF}\big(\Sigma(K)\big)$.

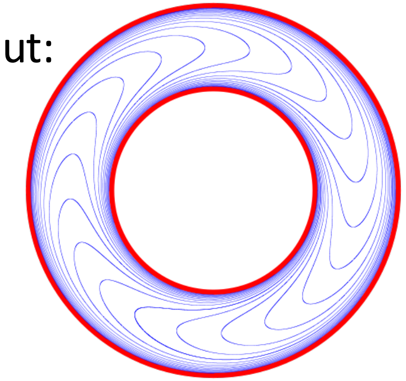
The L-space = left orderability conjecture

- **Definition**. A group G is *left orderable* if there is a total order < on G so that for all f, $g, h \in G$, $g < h \Rightarrow fg < fh$.
- **Example**. Z is left orderable.
- **Example**. Any finite group is not left orderable.
- By convention, the trivial group is not left orderable.
- **Conjecture**. (Ozsváth-Szabó, Boyer-Gordon-Watson) For irreducible rational homology spheres Y, TFAE:
  - Y is an L-space.
  - $\pi_1(Y)$ is not left orderable.
  - Y does not admit a co-orientable taut foliation.



© Ken Baker

Not taut:



Reeb foliation
Irina Gelbukh / Wikipedia

Evidence for the conjecture

- **Theorem**. (Ozsváth-Szabó 2003) If Y admits a co-orientable taut foliation then Y is not an L-space.
- **Theorem**. (Levine-Lewallen 2011) *Strong L-spaces* are not left orderable.
- Many computations. e.g., Boyer-Clay 2015, Hanselman-Rasmussen-Rasmussen-Watson 2015: true for all graph manifolds.
- Obvious: if $\tilde{Y} \to Y$ is a finite cover and $\pi_1(\tilde{Y})$ is not left orderable then $\pi_1(Y)$ is not left orderable.
- **Theorem**. (Lidman-Manolescu, 2016) If $\tilde{Y} \to Y$ is a regular, solvable cover, $\tilde{Y}$ is an L-space, and $\widehat{HF}(Y')$ is torsion free for all $\tilde{Y} \to Y' \to Y$ then Y is an L-space.
- (**Question**. Is $\widehat{HF}(Y')$ torsion free for all rational homology spheres Y?)

More L-space questions

- **Conjecture**. (Lidman, Moore) If $\Sigma(Y, K)$ is an L-space then Y is an L-space.
  - Compatible with the L-space / non-left orderable conjecture.
  - Would be implied by a spectral sequence $\widehat{HF}\big(\Sigma(Y, K)\big) \Rightarrow \widehat{HF}(Y)$.
- **Question**. (Alishahi, Levine, Lidman, …) Does $\widehat{HF}$ say anything about positive degree maps?
  - Maybe if there is a positive degree map $Y \to Y'$ then
$$\operatorname{rank}\left(\widehat{HF}(Y)\right) \geq \operatorname{rank}\left(\widehat{HF}(Y')\right)?$$
- **Question**. What can you say about relationship between $\widehat{HF}$ and π_1 beyond L-spaces?

The Poincaré conjecture

- **Theorem**. (Perelman, 2003) If $\pi_1(Y) = \{1\}$ then $Y \cong S^3$.
- **Conjecture**. (Ozsváth-Szabó) If $\widehat{HF}(Y) \cong \boldsymbol{Z}$ then Y is a connect sum of copies of the Poincaré homology sphere (and its mirror).
- Together with L-space = not left orderable conjecture, this implies the Poincaré conjecture:
  - $\pi_1(Y) = \{1\}$ (trivially) implies Y not left orderable.
  - By first conjecture, Y not left orderable implies $\widehat{HF}(Y) \cong \boldsymbol{Z}$.
  - By second conjecture, Y is a connect sum of Poincaré spheres.
  - Thus, since $\pi_1(Y) = \{1\}$, $Y = S^3$.

Thanks!

(Lots more to figure out…)