The Alexander polynomial and knot Floer homology

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(Mostly work of other people. Partly joint with Peter Ozsváth and Dylan Thurston, or David Treumann, or Kristen Hendricks and Sucharit Sarkar.)

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Seifert surfaces

• Every knot $K$ in $S^3$ bounds an orientable surface.

• **Definition.** Minimal genus of such a surface is $g(K)$.

• **Definition.** An orientable surface with boundary $K$ is a *Seifert surface* for $K$. 
Seifert matrix and Alexander polynomial

• $F$ -- a Seifert surface for $K$.
• Let $\{\gamma_i\}_{i=1}^{2g}$ -- a basis for $H_1(F)$. 
Seifert matrix and Alexander polynomial

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- $\{\gamma_i\}_{i=1}^{2g}$ -- a basis for $H_1(F)$.
- $\gamma_i^+$ be a positive pushoff of $\gamma_i$.
- **Definition.** Seifert matrix $A = (a_{i,j})$ with $a_{i,j} = lk(\gamma_i, \gamma_j^+)$. 
- **Example:** $$\begin{pmatrix} -2 & -1 \\ 0 & -2 \end{pmatrix}$$
Seifert matrix and Alexander polynomial

- $F$ -- a Seifert surface for $K$.
- $\{\gamma_i\}^2_{i=1}$ -- a basis for $H_1(F)$.
- $\gamma_i^+$ be a positive pushoff of $\gamma_i$.
- **Definition.** Seifert matrix $A = (a_{i,j})$ with $a_{i,j} = lk(\gamma_i, \gamma_j^+)$. 
- **Example.** $\begin{pmatrix} -2 & -1 \\ 0 & -2 \end{pmatrix}$
Seifert matrix and Alexander polynomial

- \( F \) -- a Seifert surface for \( K \).
- \( \{ \gamma_i \}_{i=1}^{2g} \) -- a basis for \( H_1(F) \).
- \( \gamma_i^+ \) be a positive pushoff of \( \gamma_i \).

**Definition.** Seifert matrix \( A = (a_{ij}) \) with \( a_{ij} = lk(\gamma_i, \gamma_j^+) \).

**Example:** \( \begin{pmatrix} -2 & -1 \\ 0 & -2 \end{pmatrix} \)

**Definition.** Alexander polynomial
\[
\Delta_K(t) = \pm t^n \det(tA - A^T).
\]

**Example:**
\[
\Delta_K(t) = \pm t^n \begin{vmatrix} -2t + 2 & -t \\ -2t + 2 & 1 \end{vmatrix} = \pm t^n (-4t^2 + 7t - 4) = 4t - 7 + 4t^{-1}.
\]

**Corollary.** \( 2g(K) \geq \text{width}(\Delta_K(t)) \)
Knot Floer homology  
(Ozsváth-Szabó, Rasmussen, 2003)

- $K \subset S^3$ a knot $\rightarrow$ bigraded abelian group $\widehat{HFK}_{i,j}(K)$.
- $\sum_{i,j} (-1)^i t^j \text{rank} \left( \widehat{HFK}_{i,j}(K) \right) = \Delta_K(t)$
- Homology of a chain complex
  $\left( \overline{CFK}_{i,j}(K), \partial : \overline{CFK}_{i,j}(K) \rightarrow \overline{CFK}_{i-1,j}(K) \right)$.
- Recall: $\text{width}(\Delta_K(t))/2 \leq g(K)$.

**Theorem.** (Ozsváth-Szabó) 
$$\text{width}(\overline{HFK}_{i,j}(K)) := \max\{j \mid \overline{HFK}_{*,j} \neq 0\} = g(K).$$

- Also works for null-homologous knots in other 3-manifolds.
- There are also versions for links.
- There are other variants -- $\overline{HFK}^-(K), \overline{HFK}^+(K)$, etc.
- There are extensions to **sutured manifolds** (Juhász, Alishahi-Eftekhary).

\[
\Delta_K(t) = -t + 1 - t^{-1} \quad \Delta_K(t) = 1
\]
Definition of knot Floer homology

• Via pseudo-holomorphic curves (solutions to particular nonlinear PDE’s) in a high-dimensional auxiliary space.

• There are now combinatorial definitions (Manolescu-Ozsváth-Sarkar 2006, Manolescu-Ozsváth-Szabó-Thurston 2006, ...)

• No classical definition is known. (In particular, not the singular homology of any naturally associated space.)
Fibered knots

- A knot is *fibered* if it has an $S^1$ family of Seifert surfaces.

- **Lemma.** If $K$ is fibered, fiber $F$, monodromy $\phi$ then $\Delta_K(t)$ is the characteristic polynomial of $\phi_*: H_1(F) \to H_1(F)$.

- **Corollary.** If $K$ is fibered then $\Delta_K(t)$ is monic and $\text{width}(\Delta_K(t)) = 2g(K)$.

- **Theorem.** (Ozsváth-Szabó, Ghiggini, Ni) $\widetilde{HF}(K)$ is monic if and only if $K$ is fibered.

  (Monic means $\bigoplus_i \widetilde{HF}_{i,g(K)}(K) \cong \mathbb{Z}$.)

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Lifting the characteristic polynomial formula

• Bordered Floer homology (L-Ozsváth-Thurston):
  • Surface $F \rightsquigarrow$ dg algebra $A(F)$.
    • $A(S^2) = F_2$. $A(T^2) = \rho_1 \rho_2 / \rho_2 \rho_1 = \rho_3 \rho_2 = 0$  
      $H_*(A(\Sigma_2))$ 164 diml.
Lifting the characteristic polynomial formula

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  • Cobordism $(Y, \gamma): (F_1, pt) \rightarrow (F_2, pt) \rightsquigarrow$ dg bimodule $\overline{CFDA}(Y, \gamma)$.
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  • Composition $\xrightarrow{\sim}$ derived tensor product.

• $F_1 = -F_2$ then self gluing $(Y^\circ, K)$ has $\overline{\text{HFK}} (Y^\circ, K) = HH_* \left( \overline{\text{CFDA}}(Y, \gamma) \right)$. 
Lifting the characteristic polynomial formula

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  • Surface $F \rightsquigarrow$ dg algebra $A(F)$.
    • $A(S^2) = F_2, A(T^2)$ 10-dimensional, $H_*(A(\Sigma_2))$ 164 dimensional.
  • Cobordism $(Y, \gamma): (F_1, pt) \to (F_2, pt) \rightsquigarrow$ dg bimodule $\overline{CFDA}(Y, \gamma)$.
  • Composition $\rightsquigarrow$ derived tensor product.
  • $F_1 = -F_2$ then self gluing $(Y^\circ, K)$ has $\overline{HFK}(Y^\circ, K) = HH_* \left( \overline{CFDA}(Y, \gamma) \right)$.

• Recall: for $K$ fibered, $\Delta_K(t)$ is the characteristic polynomial of $\phi_*: H_1(F) \to H_1(F)$.

• $Y_\phi =$mapping cylinder of $\phi$ then

\[
\begin{align*}
\overline{CFDA}(Y_\phi, \gamma) & \xrightarrow{HH_*} \overline{HFK}(S^3, K) \\
\phi_*: \Lambda^*H_1(F) & \xrightarrow{\text{Trace}} \Delta_K(t)
\end{align*}
\]

L-Ozsváth-Thurston, Petkova, Hom-Lidman-Watson:
Symmetries
Periodic and Freely Periodic Knots

• $K$ is *n-periodic* if it is preserved by rotation by $2\pi/n$ around an axis.

• $K$ is *freely periodic of period $(p,q)$* if it is preserved by $(z, w) \mapsto \left( e^{2\pi i/p} z, e^{2\pi i/q} w \right)$ on $S^3 = \{ (z, w) \in \mathbb{C}^2 | |z|^2 + |w|^2 = 1 \}$

$T(4,7)$ is 7-periodic, and freely $(4,7)$ periodic.

$7_4$ is 2-periodic with *quotient* the unknot $U$. 
Edmonds’s and Murasugi’s Conditions

- **Theorem.** (Edmonds, 1984) If \( K \) is \( n \)-periodic then there is a minimal genus Seifert surface for \( K \) preserved by \( \mathbb{Z}/n\mathbb{Z} \).
- (Proof uses minimal surfaces.)
- **Corollary.** If \( \overline{K} \) is the quotient knot then \( g(K) \geq ng(\overline{K}) + (n - 1)(\lambda - 1)/2 \) (where \( \lambda \) = linking number with axis).
Edmonds’s and Murasugi’s Conditions

• **Theorem.** (Edmonds, 1984) If $K$ is $n$-periodic then there is a minimal genus Seifert surface for $K$ preserved by $\mathbb{Z}/n\mathbb{Z}$.

• (Proof uses minimal surfaces.)

• **Corollary.** If $\overline{K}$ is the quotient knot then $g(K) \geq ng(\overline{K}) + (n - 1)(\lambda - 1)/2$ (where $\lambda$ = linking number with axis).

• **Corollary.** Any nontrivial knot has finitely many periods.

• **Theorem.** (Murasugi, 1971) If $K$ is $p^r$-periodic then $\Delta_K(t) \equiv \pm t^i \Delta_{\overline{K}}(t)^n \left(\frac{1-t^\lambda}{1-t}\right)^{n-1} \pmod{p}$. 
A theorem of Hendricks

• **Theorem.** [(Hendricks, 2012) + $\varepsilon$(Hendricks-L-Sarkar 2015)] If $K$ is 2-periodic then there is a spectral sequence

$$
\text{HFL}(K \cup A) \otimes (F_2 \oplus F_2) \otimes F_2[\theta, \theta^{-1}] \Rightarrow \text{HFL}(K \cup \overline{A}) \otimes F_2[\theta, \theta^{-1}].
$$

• Proof uses Seidel-I. Smith’s quantum (P.A.) Smith Theory.

• Implies (lifts) 2-periodic cases of Edmonds’s Corollary and Murasugi’s Theorem.

• Has other implications as well.
Branched double covers

- The double cover of $S^3$ branched along $K$ is
  \[ \pi: \left( \Sigma(S^3, K), \tilde{K} \right) \rightarrow (S^3, K) \]
  - $\pi|_{\Sigma(S^3, K) \setminus \tilde{K}} : (\Sigma(S^3, K) \setminus \tilde{K}) \rightarrow (S^3 \setminus K)$ is a 2-fold cover.
  - In the normal planes to $K$, $\pi$ is $z \mapsto z^2$.

- Special cases:
  - $\Sigma(S^3, U) = (S^3, U)$.
  - If $(K, A)$ is 2-periodic then $(S^3 \setminus K, A) = \Sigma(S^3 \setminus \overline{K}, \overline{A})$.

- Smith Conjecture (proved in 1980’s) states that if $K \neq U$ then $\Sigma(S^3, K) \neq S^3$.

- Can also talk about $\Sigma(Y^3, K)$, but not always unique.
Knot Floer homology in branched double covers

- **Classical?**: For $(\Sigma(K), \tilde{K}) \to (S^3, K)$, $\Delta_{\tilde{K}}(t) \equiv \Delta_K(t) \pmod{2}$.
- **Theorem.** (Hendricks, 2011) There is a spectral sequence
  \[ \overline{HF}_K(\Sigma(S^3, K), \tilde{K}) \otimes \mathbf{F}_2[\theta, \theta^{-1}] \Rightarrow \overline{HF}_K(S^3, K) \otimes \mathbf{F}_2[\theta, \theta^{-1}] \].
- **Theorem.** (L-Treumann, 2012) If $H_1(Y) = 0$ and $K \subset Y$ has $g(K) \leq 2$ then there is a spectral sequence
  \[ \overline{HF}_K(\Sigma(Y, K), \tilde{K}) \otimes \mathbf{F}_2[\theta, \theta^{-1}] \Rightarrow \overline{HF}_K(Y, K) \otimes \mathbf{F}_2[\theta, \theta^{-1}] \].
Proof sketch via bordered Floer.

• Cut $Y$ along Seifert surface $F$ for $K$ to get cobordism $Z$ from $F$ to $F$.
• $(S^3, K)$ is self-gluing of $Z$. $(\Sigma(K), \tilde{K})$ is self-gluing of $Z \cup_F Z$.
• So, want:
  \[
  \widehat{HFK}(\Sigma(K), \tilde{K}) = HH_*(CFDA(Z) \otimes^L_{A(F)} CFDA(Z))
  \Rightarrow HH_*(CFDA(Z)) = \widehat{HFK}(S^3, K)
  \]
• (Suppressed copies of $F_2[\theta, \theta^{-1}]$.)
• Such a spectral sequence exists whenever $A(F)$ satisfies certain algebraic properties.
Honest covers

• If \( \tilde{Y} \to Y \) is a \( \mathbb{Z} \) cover then there is an induced \( \mathbb{Z}/2\mathbb{Z} \) cover
  \( (\tilde{Y}/2\mathbb{Z}) = \tilde{Y} \to Y \)

• **Theorem.** (L-Treumann, 2012) If \( \tilde{Y} \to Y \) is a \( \mathbb{Z}/2\mathbb{Z} \) cover induced by a \( \mathbb{Z} \) cover then \( \widehat{HF}(\tilde{Y}) \otimes (F_2 \oplus F_2) \otimes F_2[\theta, \theta^{-1}] \Rightarrow \widehat{HF}(Y) \otimes F_2[\theta, \theta^{-1}] \).

• **Theorem.** (Lidman-Manolescu, 2016) If \( \tilde{Y} \to Y \) is a \( \mathbb{Z}/2\mathbb{Z} \) cover and \( H^1(Y) = 0 \) then for each \( s \in spin^c(Y) \),
  \( \widehat{HF}(\tilde{Y}, \pi^*s) \otimes F_2[\theta, \theta^{-1}] \Rightarrow \widehat{HF}(Y, s) \otimes F_2[\theta, \theta^{-1}] \).

• **Corollary.** \( \text{rank} \left( \widehat{HF}(\tilde{Y}, \pi^*s) \right) \geq \text{rank} \left( \widehat{HF}(Y, s) \right) \).

• Lidman-Manolescu is uses a variant of Seiberg-Witten Floer homology. Main work is identifying this variant with more common (and computable) ones.

• Some applications of Lidman-Manolescu’s result later in the talk.
Symmetric Unions

- **Theorem.** (Kinoshita-Terasaka, 1957) If \( n \) is even then \( \Delta_{K_n(J)}(t) = (\Delta_J(t))^2 \).

- **Theorem.** (Allison Moore, 2015) If replacing \( T_n \) with \( T_\infty \) gives a 2-component unlink then \( \overline{HFK}(K_n(J)) \cong \overline{HFK}(J) \otimes \overline{HFK}(m(J)) \).

- **Corollary.** \( g(K_n(J)) = 2g(J) \) and \( K_n(J) \) is fibered iff \( J \) is.

- **Corollary.** New examples satisfying *cosmetic crossing conjecture*: any non-nugatory crossing change changes isotopy class of \( K \).
More Alexander polynomial formulas

• Symmetric knots (K-T): $\Delta_{K_n}(t) = (\Delta_J(t))^2$.

• If $\Sigma^n(K)$ is n-fold cyclic branched cover of $S^3$ then

$$|H_1(\Sigma^n(K))| = \prod_{j=0}^{n-1} \Delta_K(e^{2\pi ij/n}).$$

  e.g., $|H_1(\Sigma(K))| = \Delta_K(1)\Delta_K(-1)$.

• If $\left(\Sigma^n(K), \tilde{K}\right)$ n-fold cyclic branched cover of $(S^3, K)$ then

$$\Delta_{\tilde{K}}(t^n) = \prod_{j=0}^{n-1} \Delta_K\left(e^{2\pi ij/n} t\right).$$

  e.g., if $n = 2$, $\Delta_{\tilde{K}}(t^2) = \Delta_K(t)\Delta_K(-t)$.

• (Murasugi) If $\tilde{K}$ is n-periodic, axis $\tilde{A}$, quotient $(K, A)$, $\lambda = lk(K, A)$ then

$$\frac{1-t^\lambda}{1-t} \Delta_{\tilde{K}}(t) = \prod_{j=0}^{n-1} \Delta_{K\cup A}\left(t, e^{\frac{2\pi ij}{n}}\right).$$

• (Hartley, 1981) Similar result for freely periodic knots.

• **Question.** Can any of these be lifted to Floer homology? How?
More open questions about Floer homology, Seifert surfaces, and symmetries

• There are nice combinatorial definitions of $\widehat{HFK}(S^3, K)$. Can one give combinatorial proofs of key properties, like detecting $g(K)$ or fiberedness?

• Edmonds’s condition says a minimal genus Seifert surface is $\mathbb{Z}/n\mathbb{Z}$ equivariant. Can one prove this with Floer homology? (Recall that Hendricks proved the main numerical corollary.)

• For any knot $K$ in a homology sphere $Y$, is there a spectral sequence $\widehat{HF}(\Sigma(K)) \Rightarrow \widehat{HF}(Y)$? (cf. Smith conjecture.)
Concordance
The concordance group and slice genus

• **Definition.** The *smooth slice genus* of $K$ is
  
  $$g_4(K) = \min \{ g(F) \mid F \subset D^4, \partial F = K \subset S^3 = \partial D^4, F \text{ smooth} \}.$$
The concordance group and slice genus

- **Definition.** The smooth slice genus of $K$ is
  
  $$g_4(K) = \min\{g(F)|F \subset D^4, \partial F = K \subset S^3 = \partial D^4, F \text{ smooth}\}.$$

  - $K$ is slice if $g_4(K) = 0$.

- **Definition.** The smooth concordance group is
  
  $$\mathcal{C} = ([knots]/\{slice\,\text{knots}\}, \#).$$

- Replace “smooth” by “topologically locally flat” gives topological slice genus $g_4^{\text{top}}(K)$, topologically slice knots, topological concordance group $\mathcal{C}_{\text{Top}}$.

  - $g_4^{\text{top}}(K) \leq g_4(K)$. so smoothly slice implies topologically slice.

  - $0 \to \mathcal{C}_{\text{JS}} \to \mathcal{C} \to \mathcal{C}_{\text{Top}} \to 0$. 

David Eppstein
“Square Ribbon Knot”
Wikipedia
Two classical restrictions

• Recall: the Seifert matrix $A = (a_{i,j})$ with $a_{i,j} = lk(\gamma_i, \gamma_j^+)$.  

• **Definition.** The signature $\sigma(K) \in \mathbb{Z}$ is the signature of $A + A^T$.  

• **Theorem.** (Kauffman-Taylor, Murasugi, Tristram, 1960’s) The signature is a homomorphism $C_{\mathcal{T}_{op}} \to \mathbb{Z}$, and $2g_{4}^{\text{top}}(K) \geq \sigma(K)$.  

• **Theorem.** (Fox-Milnor, 1966) If $K$ is topologically slice then there is a polynomial $f(t)$ so that $\Delta_K(t) = f(t)f(t^{-1})$.  

• **Theorem.** (Levine, 1969) There is a surjection $C_{\mathcal{T}_{op}} \to \mathbb{Z}^\infty \oplus (\mathbb{Z}/2\mathbb{Z})^\infty \oplus (\mathbb{Z}/4\mathbb{Z})^\infty$.  

• **Question.** Is there any torsion other than 2-torsion in $C$ or $C_{\mathcal{T}_{op}}$?
Some modern concordance homomorphisms

• Turning to smooth concordance:
• **Theorem.** (Endo, 1995) There is a $\mathbb{Z}^\infty$ subgroup of $C_{JS} = \ker(C \to C_{op})$.

• Does it split? Surjections onto $\mathbb{Z}$? Bounds on $g_4$?
  • Ozsváth-Szabó, 2003. $\tau: C \to \mathbb{Z}$, via Heegaard Floer homology. Bounds $g_4$.
  • Rasmussen, 2004. $s: C \to \mathbb{Z}$, via Khovanov homology. Bounds $g_4$.
  • Manolescu-Owens, 2005. $\delta: C \to \mathbb{Z}$, via Heegaard Floer homology.
  • Livingston, 2006. $\tau, s, \delta$ give a surjection $C_{JS} \to \mathbb{Z}^3$.
  • Hom, 2013. A surjection $C_{JS} \to \mathbb{Z}^\infty$. (Not constructive.)
  • Ozsváth-Stipsicz-Szabó, 2014. Another surjection $\Upsilon: C_{JS} \to \mathbb{Z}^\infty$. (Constructive.)
  • Hendricks-L-Sarkar, 2015. Another homomorphism $q_\tau: C \to \mathbb{Z}$, via equivariant Floer homology.

• Many other beautiful results by many other researchers.
An explicit example in $\mathcal{C}_{JS}$

- Recall: $0 \to \mathcal{C}_{JS} \to \mathcal{C} \to \mathcal{C}_{top} \to 0$.

- **Theorem.** (Freedman-Quinn ~1989) If $\Delta_K(t) = 1$ then $K$ is topologically slice.

- **Exercise.** The (positive) untwisted Whitehead double $D_+(K)$ of any knot $K$ has $\Delta_{D_+(K')}(t) = 1$.

- **Theorem.** (Hedden 2006) If $\tau(K) > 0$ then $\tau(D_+(K)) = 1$.

- **Corollary.** $\mathcal{C}_{JS}$ is nontrivial.

- (This is not the first proof, which is attributed to Casson (unpublished). Gompf gave first examples of non-slice Whitehead doubles.)
From $C_{JS}$ to exotic $R^4$'s

- **Theorem.** (Classical, Moise-Munkres ~1960, Stallings 1961) There is a unique smooth structure on $R^n$ is $n \leq 2$, $n = 3$, $n > 4$.

- **Theorem.** (Taubes, 1987) There are uncountably many smooth structures on $R^4$.

- **Theorem.** (Freedman-Quinn) Any non-compact 4-manifold has a smooth structure.

Here’s an exotic $R^4$ (Gompf-Stipsicz, Exercise 9.4.23):

- Take a knot $K$ in $C_{JS}$. Let $X_K$ be result of attaching a 2-handle to $D^4$ along $K$. There is a topologically flat embedding $X_K \hookrightarrow R^4$.

- Freedman-Quinn implies $R^4 \setminus X_K$ can be smoothed. Glue $X_K$ to this smoothing of $R^4 \setminus X_K$ to get a smooth $R^4$. Must be exotic, because $K$ is smoothly slice in it.
Lifting the Fox-Milnor condition

• Recall: the Fox-Milnor condition: if $K$ is topologically slice then $\Delta_K(t) = P(t)P(t^{-1})$.

• Question. Is there a lift of this result to $\overline{HFK}$?
  • Presumably such a result would be about smoothly slice knots.
  • The Kinoshita-Terasaka knot is topologically slice (since $\Delta_K(t) = 1$) but $\overline{HFK}(K)$ has Poincaré polynomial $(q^{-2} + q^{-1})t^{-2} + 4(q^{-1} + t)t^{-1} + 7 + 6q + 4(q + q^2)t + (q^2 + q^3)t^2$, and total rank $2 + 8 + 7 + 6 + 8 + 2 = 33$ (not square!).
  • Maybe some sort of spectral sequence?
  • Maybe only four doubly slice knots (knots with an invertible concordance to $U$)?
Some more open concordance questions

• Is there a surjection $C_{JS} \to (\mathbb{Z}/2\mathbb{Z})^\infty$? Any other torsion?

• **Conjecture.** (Kirby 1.38) $D_+ (K)$ is smoothly slice if and only if $K$ is.

• **Sub-conjecture.** The positive Whitehead double of the negative trefoil is not smoothly slice.

• What is the smooth slice genus of the Conway knot? (Its mutant, the Kinoshita-Terasaka knot, is slice. $\Delta_{Conway}(t) = \Delta_{KT}(t) = 1$.)
Dehn surgery is Ouroboros holes.

Adapted from (German) Wikipedia entry, contributed by AnonMoos.
Definition of Dehn surgery

• **Definition.** *Dehn surgery* on a link \( L = K_1 \cup \cdots \cup K_n \) is \( S^3 \setminus \text{nbd}(K_1 \cup \cdots K_n) \cup_{\partial} (D^2 \times S^1 \cup \cdots \cup D^2 \times S^1) \).

• \( \partial(S^3 \setminus \text{nbd}(K_1 \cup \cdots K_n)) = T^2 \sqcup \cdots \sqcup T^2 \). Surgery is determined by images of the \( n \) circles \( \partial D^2 \times \{ pt \} \) which bound disks \( D^2 \).

• Each circle is \( p \cdot \text{(meridian)} + q \cdot \text{(longitude)} \), so have \( S_{\frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n}}^3(L) \).

• **Theorem.** (Lickorish, 1962) Every closed, orientable 3-manifold is \( S_{\frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n}}^3(L) \) for some \( \frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n} \) and \( L \).
Some questions and some answers

• **Question.** Which manifolds arise as surgery on a knot?
  • There are some obvious restrictions on $H_1, \pi_1$.
  • There are non-obvious restrictions on manifolds with nontrivial first homology (Boyer-Lines 1990).

• **Theorem.** (Hom-Karakurt-Lidman, 2014) There are non-obvious restrictions on manifolds with $H_1 = 0$. e.g.,
  $$\Sigma(p, 2p - 1, 2p + 1) = \left\{ (z_1, z_2, z_3) \in S^5 \subset \mathbb{C}^3 \mid z_1^p + z_2^{2p-1} + z_3^{2p+1} = 0 \right\},$$
  $p \geq 8$ does not.

• **Question.** Are there non-obvious restrictions on which manifolds arise as surgery on an $n$-component link ($n>1$ fixed)?
More questions and answers

• **Question.** If $S^3_{\frac{p}{q}}(K) \cong S^3_{\frac{p}{q}}(K')$ must $K = K'$?

  - **Theorem.** (Gabai, Kronheimer-Mrowka-Ozsváth-Szabó, 2003) Yes if $K=U$.
  - **Theorem.** (Ghiggini, Ozsváth-Szabó, 2006) Yes if $K = 3_1, m(3_1)$, or $4_1$. 

Knot Atlas / KnotPlot
Surgery characterizations of knots

• **Question.** If $S^3_p(K) \cong S^3_p(K')$ must $K = K'$?
  • **Theorem.** (Gabai, Kronheimer-Mrowka-Ozsváth-Szabó, 2003) Yes if $K=U$.
  • **Theorem.** (Ghiggini, Ozsváth-Szabó, 2006) Yes if $K = 3_1, m(3_1), \text{ or } 4_1$.
  • “No” in general.
  • **Question.** What about other specific knots (e.g., $T_{3,4}$)?
Lens space surgeries

• **Question.** For which $K$ is there $p/q$ with $S_{p/q}^3(K)$ a lens space?

• **Berge Conjecture.** Such $K$ is *doubly primitive*, i.e.,
  • Lies on a genus 2 Heegaard surface
  • Meets some meridian disk on each side once.

• **Theorem.** (Ni, 2006): If $S_{p/q}^3(K)$ is a lens space then $K$ is fibered.

• **Theorem.** (Greene, 2010): The lens spaces that arise from knot surgery are those predicted by Berge conjecture.
An application of equivariant Floer homology

• **Question.** When does one surgery on a knot cover another?

• **Example.** \( S_{6q+1}^3 (3_1) = L(6q + 1, 4q) \) (Moser, 1971). So \( S_{6q+1}^3 (3_1) \) is a \((6k + 1)\)-fold cover of \( S_{6q'+1}^3 (3_1) \) for \( q' = q + k(6k + 1) \).

• **Theorem.** (Lidman-Manolescu, 2016) If \( \frac{p}{q} < 1 \) and \( \left[ \frac{q}{p} \right] < \left[ \frac{q'}{p'} \right] \) then \( S_p^q (K) \) is not a regular \( r^n \) sheeted cover of \( S_{p'}^{q'} (K) \) (\( r \) prime).

• (Proof uses equivariant Floer homology spectral sequence \( \widehat{HF} (\tilde{Y}) \rightarrow \widehat{HF} (Y) \) mentioned earlier.)
L-spaces and left orderability
## L-spaces

- **Definition.** $Y^3$ is a *rational homology sphere* if $H_*(Y; \mathbb{Q}) \cong H_*(S^3, \mathbb{Q})$.

- **Theorem.** (Ozsváth-Szabó) If $Y^3$ is a rational homology sphere then
  \[ \chi(\widehat{HF}(Y)) = |H_1(Y)| = |\text{spin}^C(Y)|. \]

- **Definition.** A rational homology sphere $Y^3$ is an *L-space* if $\widehat{HF}(Y) \cong \mathbb{Z}^{\mid H_1(Y)\mid}$ (i.e., as small as possible).

- **Examples.**

<table>
<thead>
<tr>
<th>Space</th>
<th>$H_1$</th>
<th>$\widehat{HF}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Poincaré homology sphere $S^3_1(3_1)$</td>
<td>0</td>
<td>$\mathbb{Z}$</td>
</tr>
<tr>
<td>Lens space $L(p, q) = S^3_p(U)$ $\frac{q}{\bar{q}}$</td>
<td>$\mathbb{Z}/p\mathbb{Z}$</td>
<td>$\mathbb{Z}^p$</td>
</tr>
<tr>
<td>(Any manifold with spherical geometry.)</td>
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<tr>
<td>$\Sigma(K)$ for $K$ alternating</td>
<td>$</td>
<td>H_1</td>
</tr>
<tr>
<td>$S^3_p(P(-2,3,7))$ (Hyperbolic for $p&gt;19$)</td>
<td>$\mathbb{Z}/p\mathbb{Z}$</td>
<td>$\mathbb{Z}/p\mathbb{Z}$</td>
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</tbody>
</table>

(From Wikipedia)
Some results for L-spaces

• Strangely, Floer theory is very useful for studying L-spaces. Some examples:

  • **Theorem.** (Ozsváth-Szabó 2003) If $S^3_p(K)$ is an L-space then
    \[ \Delta_K(t) = (-1)^k + \sum_{j=1}^{k}(-1)^{k-j}(T^{n_j} + T^{-n_j}) \]
    for an increasing sequence $0 < n_1 < \cdots < n_k$.
    • Strong restriction on what knots have lens space surgeries.

  • **Theorem.** (Lidman-Moore 2015) If $\Sigma(K)$ is an L-space and $\det(K)$ is square-free then $K$ satisfies the cosmetic crossing conjecture.

  • **Theorem.** (Greene 2011) If $K, K'$ are alternating and $\Sigma(K) \cong \Sigma(K')$ then $K$ and $K'$ are related by a sequence of mutations.
    • Partial converse to an old observation of Viro.
    • Uses (grading on) $\widehat{H}F(\Sigma(K))$. 

The L-space = left orderability conjecture

• **Definition.** A group $G$ is *left orderable* if there is a total order $<$ on $G$ so that for all $f, g, h \in G$, $g < h \Rightarrow fg < fh$.

• **Example.** $\mathbb{Z}$ is left orderable.

• **Example.** Any finite group is not left orderable.

• By convention, the trivial group is not left orderable.

• **Conjecture.** (Ozsváth-Szabó, Boyer-Gordon-Watson) For irreducible rational homology spheres $Y$, TFAE:
  • $Y$ is an L-space.
  • $\pi_1(Y)$ is not left orderable.
  • $Y$ does not admit a co-orientable taut foliation.
Evidence for the conjecture

• **Theorem.** (Ozsváth-Szabó 2003) If $Y$ admits a co-orientable taut foliation then $Y$ is not an L-space.

• **Theorem.** (Levine-Lewallen 2011) *Strong L-spaces* are not left orderable.

• Many computations. e.g., Boyer-Clay 2015, Hanselman-Rasmussen-Rasmussen-Watson 2015: true for all graph manifolds.

• Obvious: if $\tilde{Y} \to Y$ is a finite cover and $\pi_1(\tilde{Y})$ is not left orderable then $\pi_1(Y)$ is not left orderable.

• **Theorem.** (Lidman-Manolescu, 2016) If $\tilde{Y} \to Y$ is a regular, solvable cover, $\tilde{Y}$ is an L-space, and $\widehat{HF}(Y')$ is torsion free for all $\tilde{Y} \to Y' \to Y$ then $Y$ is an L-space.

• *(Question. Is $\widehat{HF}(Y')$ torsion free for all rational homology spheres $Y$?)*
More L-space questions

• **Conjecture.** (Lidman, Moore) If $\Sigma(Y, K)$ is an L-space then $Y$ is an L-space.
  • Compatible with the L-space / non-left orderable conjecture.
  • Would be implied by a spectral sequence $\widehat{HF}(\Sigma(Y, K)) \Rightarrow \widehat{HF}(Y)$.

• **Question.** (Alishahi, Levine, Lidman, ...) Does $\widehat{HF}$ say anything about positive degree maps?
  • Maybe if there is a positive degree map $Y \to Y'$ then
    \[
    \text{rank} \left( \widehat{HF}(Y) \right) \geq \text{rank} \left( \widehat{HF}(Y') \right) ?
    \]

• **Question.** What can you say about relationship between $\widehat{HF}$ and $\pi_1$ beyond L-spaces?
The Poincaré conjecture

• **Theorem.** (Perelman, 2003) If \( \pi_1(Y) = \{1\} \) then \( Y \cong S^3 \).

• **Conjecture.** (Ozsváth-Szabó) If \( \widehat{HF}(Y) \cong \mathbb{Z} \) then \( Y \) is a connect sum of copies of the Poincaré homology sphere (and its mirror).

• Together with L-space = not left orderable conjecture, this implies the Poincaré conjecture:
  • \( \pi_1(Y) = \{1\} \) (trivially) implies \( Y \) not left orderable.
  • By first conjecture, \( Y \) not left orderable implies \( \widehat{HF}(Y) \cong \mathbb{Z} \).
  • By second conjecture, \( Y \) is a connect sum of Poincaré spheres.
  • Thus, since \( \pi_1(Y) = \{1\}, \ Y = S^3 \).
Thanks!

(Lots more to figure out...)