Putting bordered Floer homology in its place: a contextualization of an extension of a categorification of a generalization of a specialization of Whitehead torsion

Robert Lipshitz, Joint with Peter Ozsváth and Dylan Thurston

April 4, 2009

Review of classical Heegaard Floer homology (Ozsváth-Szabó theory)

To a	Heegaard Floer assigns
Closed Y^3, $\mathfrak{s} \in \text{spin}^c(Y)$	Groups $\widehat{HF}(Y,\mathfrak{s}) = H_*(\widehat{CF}(Y,\mathfrak{s}))$, $HF^-(Y,\mathfrak{s}) = H_*(CF^-(Y,\mathfrak{s}))$,...
$(W^4,\mathfrak{t})$, $\partial W^4 = -Y_1 \cup Y_2$, $\mathfrak{t} \in \text{spin}^c(W^4)$	Maps $\hat{F}_{W,\mathfrak{t}}\colon \widehat{HF}(Y_1,\mathfrak{t}\|_{Y_1}) \to \widehat{HF}(Y_2,\mathfrak{t}\|_{Y_2})$, ...

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- Heegaard Floer carries lots of topological information.

Goals of bordered Floer homology

Extend the Heegaard Floer invariant $\widehat{HF}$ to 3-manifolds with boundary in a way that is:

- Simple enough to be computable in examples and
- Contains enough information to recover the closed invariant.

Conventions and caveats

- Coefficients will be $\mathbb{Z}/2$ unless otherwise specified
- Theorems about bordered HF are joint work (sometimes in progress) with Peter Ozsváth and Dylan Thurston.
- To be efficient, I will tell some lies.
- I apologize if I miss relevant work.

Next we will discuss the. . .

1. Basic structure of bordered HF
2. Bimodules and reparametrization
3. Self-gluing and Hochschild Homology
4. Other extensions of Heegaard Floer

Bordered HF assigns. . .

To a F^2 closed, oriented	bordered HF assigns *dg* algebra $\mathcal{A}(F)$
Y^3, $\partial Y = F$	Left *dg* $\mathcal{A}(-F)$-module $\widehat{CFD}(Y)$ Right $\mathcal{A}_\infty$ $\mathcal{A}(F)$-module $\widehat{CFA}(Y)$.

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If $Y = Y_1 \cup_F Y_2$ *then* $\widehat{CF}(Y) \simeq \widehat{CFA}(Y_1) \tilde{\otimes}_{\mathcal{A}(F)} \widehat{CFD}(Y_2)$.

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Interpreting this in the bordered way...

- $\mathcal{A}(S^2) \simeq \mathbb{Z}/2$.
- $\widehat{CFA}(Y \setminus \mathbb{D}^3) \cong \widehat{CFD}(Y \setminus \mathbb{D}^3) \cong \widehat{CF}(Y)$.

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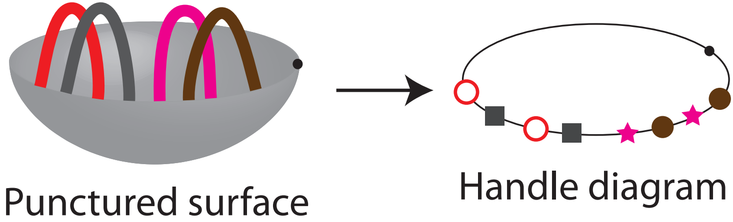
- $\widehat{CFA}(Y \setminus \mathbb{D}^3) \cong \widehat{CFD}(Y \setminus \mathbb{D}^3) \cong \widehat{CF}(Y)$.
- Note that with $\mathbb{Z}$-coefficients, working at the chain level avoids Tor-terms.

A picture of $\mathcal{A}(F)$.



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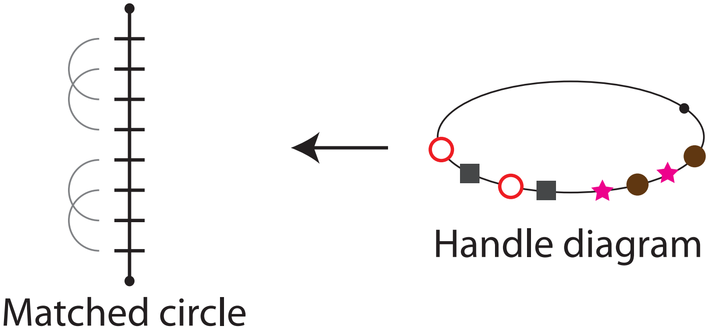

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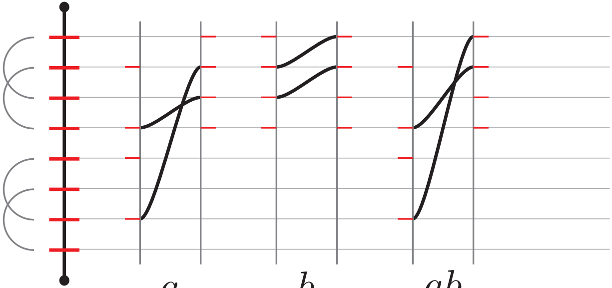

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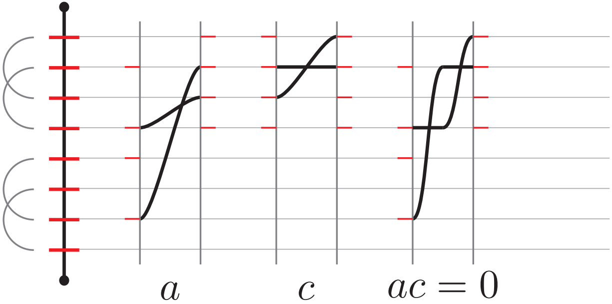

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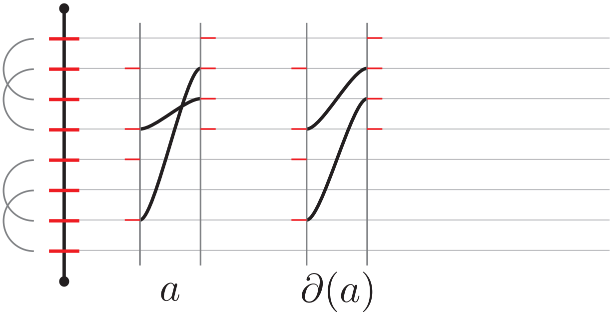

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Special cases...

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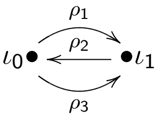
- 

$$/(\rho_2\rho_1 = \rho_3\rho_2 = 0).$$

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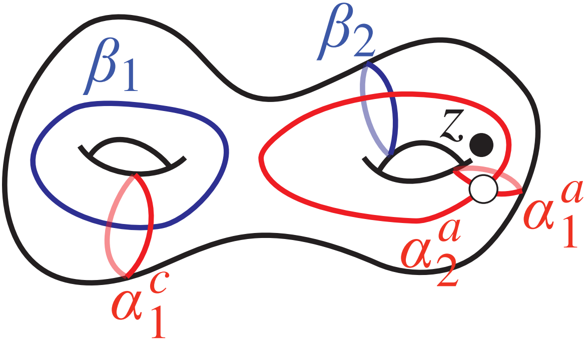

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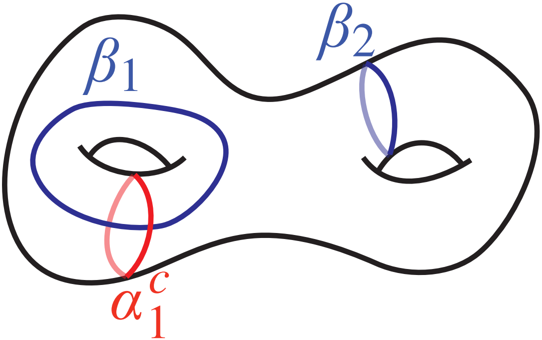

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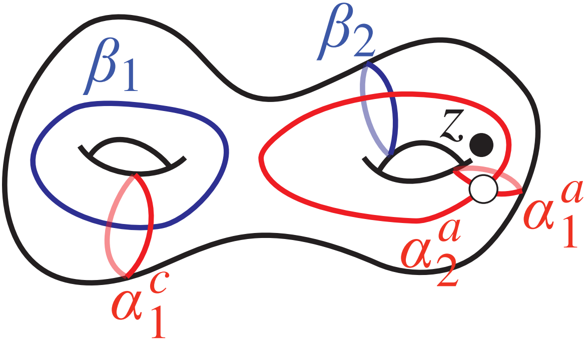

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- Treating the boundary as a puncture, count curves in $\Sigma \times [0,1] \times \mathbb{R}$.
- Boundary is a pointed matched circle. Asymptotics at boundary give algebra elements.
- Interesting curves contribute to the differential ($\widehat{CFD}$) or the module structure ($\widehat{CFA}$).

Segal's notion of a $(n+1+1)$-dimensional TQFT

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F^2 closed	$\longrightarrow$	category $\mathcal{C}(F)$ enriched over vector spaces
Y^3, $\partial Y = F$	$\longrightarrow$	element $Z(Y) \in \mathcal{C}(F)$

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Such that: $Z(Y_1 \cup_F -Y_2) = \mathrm{Hom}(Z(Y_1), Z(Y_2))$
(plus more axioms).

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- Hom is like tensor product. In fact...

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(in progress) $\widehat{CF}(Y_1 \cup_F Y_2) \simeq \mathrm{RHom}(\widehat{CFD}(Y_1), \widehat{CFD}(-Y_2)) \simeq \mathrm{RHom}(\widehat{CFA}(-Y_1), \widehat{CFA}(Y_2))$.

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(This follows from the fact that $\widehat{CFD}(Y)$ is *Koszul dual* to $\widehat{CFA}(-Y)$.)

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$$\begin{array}{ccc} n \in \mathbb{N} & \longrightarrow & \text{algebra } H^n \\ (m,n)\text{-tangle } T & \longrightarrow & \text{complex of } (H^n, H^m)\text{-bimodules } \mathcal{F}(T). \end{array}$$

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Definition

The *nilCoxeter algebra* A_n has generators $Y_1, \dots, Y_{n-1}$ and relations $Y_i^2 = 0$; $Y_i Y_j = Y_j Y_i$ if $|i-j| > 1$; and $Y_i Y_{i+1} Y_i = Y_{i+1} Y_i Y_{i+1}$.

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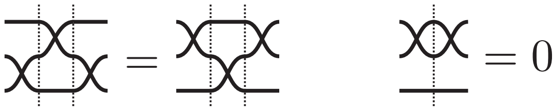

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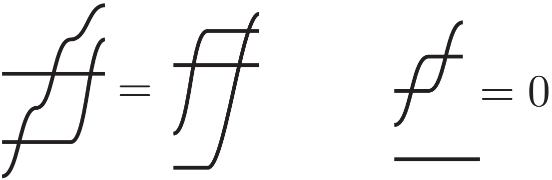

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- appears in Khovanov-Lauda's categorification of $U_q(\mathfrak{sl}_n)$.

The next level of structure is. . .

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- (More generally, for any Y with two boundary components there is an associated bimodule $\widehat{CFDA}(Y)$.)

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Theorem

(Khovanov) Associated to a braid B on n strands is a complex of (H^n, H^n)-bimodules $\mathcal{F}(B)$ such that for any tangle T,

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More group actions on categories

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- actions on categories of sheaves in geometric representation theory (Rouquier, Seidel-Thomas, Stroppel,. . .)

(See intro to Khovanov-Thomas "Braid cobordisms, triangulated categories and flag varieties" for references.)

The Wehrheim-Woodward machinery

- Heegaard Floer was originally defined via Lagrangians in $\mathrm{Sym}^g(\Sigma)$. That is:

Construction

To a handlebody $\mathcal{H}$, $\partial\mathcal{H} = \Sigma$ is associated an element $T(H) \in \mathrm{Fuk}(\mathrm{Sym}^g(\Sigma))$.

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Surface F	$\longrightarrow$	$\mathrm{Fuk}(\mathrm{Sym}^g(F))$
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- (This has not been carried-out.)

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- Heegaard Floer was originally defined via Lagrangians in $\mathrm{Sym}^g(\Sigma)$. That is:

Construction

To a handlebody $\mathcal{H}$, $\partial\mathcal{H} = \Sigma$ is associated an element $T(H) \in \mathrm{Fuk}(\mathrm{Sym}^g(\Sigma))$.

$$\widehat{CF}(H_1 \cup_F H_2) \cong \mathrm{Hom}_{\mathrm{Fuk}(\mathrm{Sym}^g(\Sigma))}(T(H_1), T(H_2)).$$

- Wehrheim-Woodward's theory of quilted Floer theory indicates how to extend this:

$$\begin{array}{ccc} \text{Surface } F & \longrightarrow & \mathrm{Fuk}(\mathrm{Sym}^g(F)) \\ Y^3,\ \partial Y = -F_1 \cup F_2 & \longrightarrow & \textit{Lagrangian correspondence} \\ & & \text{from } \mathrm{Sym}^{g_1}(F_1) \text{ to } \mathrm{Sym}^{g_2}(F_2). \end{array}$$

Question

How does this story relate to bordered Floer homology?

Reza Rezazadegan's extension of Kh_{symp} to tangles

- Seidel-Smith introduced a knot invariant Kh_{symp} via symplectic geometry which is:
  - conjectured to agree with Khovanov homology and
  - defined similarly to Heegaard Floer homology.

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Question

Is there an analogue of bordered Floer theory for Kh_{symp}? If so, how does it relate to Rezazadegan's theory?

We have made it as far as. . .

In bordered HF

- Fix $p \in F$, $v \in T_pF$. Then $MCG_0 = \{\phi\colon (F, p, v) \to (F, p, v)\}/\sim$.

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- For $\phi \in MCG_0(F)$, $T_\phi = [0,1] \times F/((1,x) \sim (0,\phi(x)))$.
- $K = p \times [0,1]$ is a framed knot in T_ϕ.
- Let Y_ϕ denote 0-surgery of T_ϕ along K. Then:

Theorem

Let HH_* denote Hochschild homology. Then

$$HH_*(\widehat{CFDA}(\phi)) \cong \widehat{HFK}(Y_\phi, K).$$

Analogue in Khovanov-Rozansky homology

Theorem

(Khovanov) Associated to each braid $B \in B_n$ is a complex $F(B)$ of bimodules so that the Hochschild homology $HH_*(F(B))$ is isomorphic to the Khovanov-Rozansky HOMFLY homology.

A conjecture of Kontsevich-Seidel

Conjecture

(Kontsevich[a], Seidel[b]) Let $\phi\colon F \to F$ *be a symplectomorphism, and* $HF^*(\phi)$ *its Floer cohomology.* ϕ *induces a functor* $\phi_*\colon \mathrm{Fuk}(F) \to \mathrm{Fuk}(F)$*. Then*

$$HF^*(\phi) \cong HH^*(\mathbb{I}_*, \phi_*).$$

[a] "Homological algebra of mirror symmetry"

[b] "Fukaya categories and deformations"

Warnings:

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[a] "Homological algebra of mirror symmetry"
[b] "Fukaya categories and deformations"

Warnings:

- This is not literally what they say.
- The precise definitions of the objects involved (homology of the symplectomorphism ϕ, Fukaya category) are important.

Aside: what Kontsevich actually said...

"By the general philosophy, A_∞-deformations of first order of $F(V)$ should correspond to Ext-groups in a category of functors $F(V) \to F(V)$. The natural candidate for such a category is $F(V \times V)$ where the symplectic structure on $V \times V$ is $(\omega, -\omega)$. The diagonal $V_{diag} \subset V \times V$ is a Lagrangian submanifold and it corresponds to the identity functor. By a version of Floer'es theorem (see [F]) there is a canonical isomorphism between the Floer cohomology $H^*(\mathrm{Hom}_{F(V \times V)}(V_{diag}, V_{diag})$ and the ordinary topological cohomology $H^*(V, \mathbb{C})$. ..."
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Translation: The Hochschild homology of the Fukaya category of M should be the Floer homology of the identity map $M \to M$.

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Translation: The Hochschild homology of the Fukaya category of M should be the Floer homology of the identity map $M \to M$.

(Thanks to Tim Perutz for the translation.)

...and what Seidel actually said.

Paul Seidel, "Fukaya categories and deformations":

Theorem

Suppose that M is a compact, exact symplectic manifold with contact type boundary. Then there is a natural map

$$SH^*(M) \to HH^*(\mathrm{Fuk}(M), \mathrm{Fuk}(M))$$

where $SH^*(M)$ denotes the symplectic homology of M.

Conjecture

Under appropriate conditions, the map $SH^*(M) \to HH^*(\mathrm{Fuk}(M), \mathrm{Fuk}(M))$ is an isomorphism.

(Again, thanks to Tim Perutz for pointing these out.)

More speculation on Kontsevich's conjecture

Conjecture

For $\phi\colon F \to F$ a symplectomorphism,

$$HF(\phi) \cong \widehat{HF}(T_\phi, \mathfrak{s}_{g-2}).$$

(Part of the Heegaard-Floer = Seiberg-Witten = embedded contact = periodic Floer = quilted Floer =... conjecture.)

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Pseudo-conjecture

The bordered Floer $\mathcal{A}(F)$ "is" $\bigoplus_{n=0}^{2g(F)} \mathrm{Fuk}(\mathrm{Sym}^n(F))$.

Given both conjectures, the bordered Floer self-gluing theorem would be some version of the Kontsevich-Seidel conjecture.

You have survived to. . .

Knot Floer homology

Knot Floer homology (Osváth-Szabó, Rasmussen) associates to nullhomologous knot $K \hookrightarrow Y$ filtered complexes $\widehat{CFK}(Y, K)$, $CFK^{-}(Y, K)$, ...

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Theorem

There is an $\mathcal{A}(T^2)$-module $\widehat{CFA}(S^1 \times \mathbb{D}^2, S^1 \times \{0\})$ so that

$$\widehat{CFK}(Y, K) \simeq \widehat{CFA}(S^1 \times \mathbb{D}^2, S^1 \times \{0\}) \,\widetilde{\otimes}\, \widehat{CFD}(Y)$$

Surgery theorems

Conversely...

Theorem

For $K \hookrightarrow S^3$ a knot, $\widehat{CFD}(S^3 \setminus K)$ is determined by $CFK^-(S^3, K)$.

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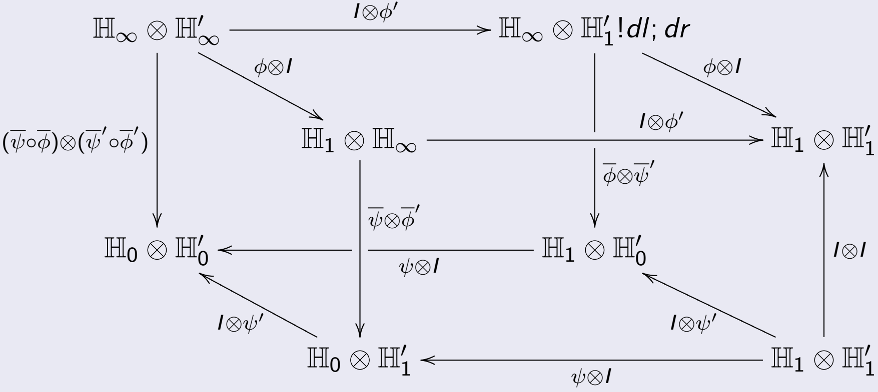
The analogue for links is not known.

Eftekhary's splicing theorems

Theorem

(Eftekhary) Let $K \hookrightarrow Y$ be a knot and $\mathbb{H}_n = \widehat{HFK}(Y_n(K), K)$. Then there are maps $\phi, \overline{\phi}\colon \mathbb{H}_\infty \to \mathbb{H}_1$ and $\psi, \overline{\psi}\colon \mathbb{H}_1 \to \mathbb{H}_0$ so that for (Y', K') another knot, $\widehat{CF}(Y_K \#_{K'} Y') \simeq$



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- The construction of these maps is similar to the construction of bordered Floer.

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- The construction of these maps is similar to the construction of bordered Floer.
- (An exact relationship between Eftekhary's theory and bordered Floer is not yet known.)

Sutured Floer homology

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Theorem

Let S *be a collection of sutures on* F^2*. There is an* $\mathcal{A}(F)$*-module* $M(S)$ *so that for any* Y *with* $\partial Y = F$,

$$SFH(Y, S) = H_*(\widehat{CFA}(Y) \tilde{\otimes} M(S)).$$

The contact element in Sutured Floer

- If (Y, ξ) is contact and ∂Y convex then ξ has a *dividing set*, a set of curves in ∂Y.

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- This raises a natural...

Question

Is there a contact element in bordered Floer homology.

Don't think about this question: I think it's already been solved (not by us).

HKM's contact category

- $SFH(Y_1)$ and $SFH(Y_2)$ is not enough information to reconstruct $\widehat{HF}(Y_1 \cup_\partial Y_2)$. But...

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Question

Does the contact category contain enough information to reconstruct bordered Floer?

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Question

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- If so. . .

The end.

- Thanks for listening.
- And thanks again to the organizers for organizing!