Some Computational Results about Grid Diagrams of Knots

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1 Introduction

From Heegaard Floer homology, an invariant for three-manifolds, one can construct an invariant for knots- knot Floer homology $\widehat{HFK}$. $\widehat{HFK}$ is a finite dimensional bi-graded vector space over $\mathbb{Z}/2$. One can introduce as well $\widetilde{HFK}$, which is an invariant of grid diagrams dependent only on the knot K and the arc index n of the diagram. It is a theorem from [MOS06] that there is an isomorphism $\widetilde{HFK} = \widehat{HFK} \bigotimes V^{n-1}$ where V is a two dimensional $\mathbb{Z}/2$ vector space. Thus from $\widetilde{HFK}$ we can recover $\widehat{HFK}$ which is a knot invariant independent of n. $\widetilde{HFK}$ is related to the Alexander polynomial $\Delta_K(T)$ by the formula:

$$(1 - t^{-1})^{n-1}\Delta_K(T) = \sum_{i,j}(-1)^j t^i \dim \widetilde{HFK}_{i,j} \qquad (1)$$

From [MOST06] it can be seen that $\widehat{HFK}$ can be arrived at by combinatorial means. We can do this by working with toroidal grid diagrams. A toroidal grid diagram is a planar grid diagram where the top and bottom edges are identified and the left and right edges are identified. A planar grid diagram is a $n \times n$ grid where every row contains exactly one X and one O, every column contains exactly one X and one O and no cell contains more than one X or O.

We draw lines between X or O if they are in the same column or row. If we have a crossing we let the vertical line pass over the horizontal.

We can associate a chain complex to a toroidal grid diagram.

The generators are given by the n-tuples of intersection points between horizontal and vertical arcs (viewed as circles on the torus) on the diagram,

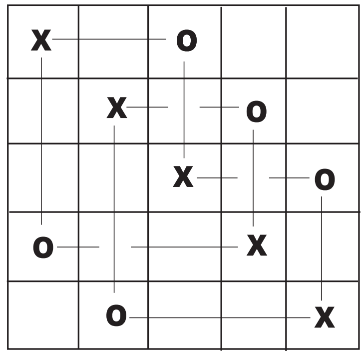

Figure 1: A Grid Diagram for the Trefoil

with the added condition that every intersection point appears on a unique horizontal and vertical circle . There is a straightforward one-to-one correspondence between generators of our complex and elements $x \in S_n$. This choice of labelling for the generators depends on how one cuts open the torus.

We define for A,B two collections of points on the plane the number $I(A,B)$. $I(A,B)$ counts the number of pairs $(a_1,a_2) \in A$ and $(b_1,b_2) \in B$ with $a_1 < b_1$ and $a_2 < b_2$.

Finally we define $J(A,B)$ as the average of numbers $I(A,B)$ and $I(B,A)$.

We define the functions $A(x)$ and $M(x)$ as follows:

$$A(x) = J(x,X) - J(x,O) - \frac{1}{2}J(X,X) + \frac{1}{2}J(O,O) - \frac{n-1}{2} \quad (2)$$

$$M(x) = J(x,x) - 2J(x,O) + J(O,O) + 1 \quad (3)$$

In our chain complex $\widetilde{CFK}$ we say the generator x $\in \widetilde{CFK}_{ij}$ if and only if A(x)=i and M(x)=j.

There is a differential in our bigraded chain complex δ: $\widetilde{CFK}_{ij} \rightarrow \widetilde{CFK}_{ij-1}$. The differential satisfies $\delta^2 = 0$.

It is the case that $\widetilde{HFK}$ is the homology of $\widetilde{CFK}$.

In this paper, we will use the combinatorial methods outlined by [MOST06] to compute the genus of all torus knots. We use the following statement

$$g(K) = \frac{(\text{highest Alexander grading in } \widetilde{HFK} \neq 0) - (\text{lowest Alexander grading in } \widetilde{HFK} \neq 0)}{2} - \frac{n-1}{2}$$

where $g(K)$ denotes the genus of a knot K. This is adapted from theorem 1.2 of [OS04] and the isomorphism $\widetilde{HFK} = \widehat{HFK} \bigotimes V^{n-1}$ from [MOST06].

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2 Lemmas about Grid Diagrams

We say that a point on a grid diagram $\hat{a} = (k, a_k)$ has a weight $w(\hat{a}) = c - d$ where c is the number of $X = (a, b)$ such that $a \geq k$, $b \geq a_k$, d is the number of Xs such that $k > a$, $a_k > b$

Lemma 1. *On any grid diagram if we have $w(\hat{a}) = r$, then $\hat{b} = (k+1, a_k)$ has weight $r-1$, similarly $\hat{c} = (k, a_k + 1)$ has weight $r-1$.*

Proof. Say $w(\hat{a}) = r$ where $\hat{a} = (k, a_k)$. So $(k+1, a_k)$ is a shift over to the right of the diagram. In the column separating the two points there exists exactly one $X = (a, b)$.There are two cases $b \geq a_k$ or $b < a_k$. In first case we get $w(k+1, a_k) = (c-1) - d = r - 1$ and in the second $w(k+1, a_k) = c - (d+1) = r - 1$. A similar argument can be made for $\hat{c}$, where the only difference in the weight must be one X in the row separating the points. □

Figure 2 gives an example of a grid diagram with the weights filled in. The distribution of the weights solely depends on the size of the grid diagram; thus two $n \times n$ grid diagrams of two different knots will have the same weight distribution.

Lemma 2. *For any grid diagram $I(x, X) - I(X, x) = n$ where $x = Id$.*

Proof. Say that the $\hat{a}_i$ are the points that make up the generator x. Then $I(x, X) - I(X, x) = w(\hat{a}_1) + w(\hat{a}_2) + ... + w(\hat{a}_n) = n + (n-2) + (n-4)... + (n - 2(n-1)) = n^2 - 2\sum_{i=1}^{n-1} i = n^2 - 2\frac{n(n-1)}{2} = n$. □

Lemma 3. *Let (ab) denote the transposition exchanging a and b. For a generator x if $I(x, X) - I(X, x) = n$ then $I((ab)x, X) - I(X, (ab)x) = n$.*

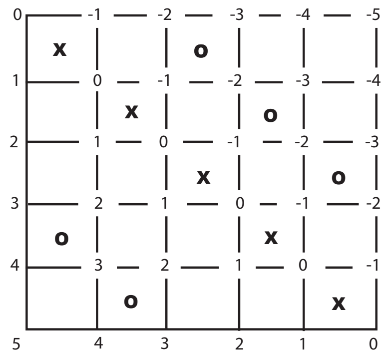

Figure 2: The weights for a trefoil

Proof. Let $y = (ab)x$. Then the intersection points that x and y do not have in common form a rectangle of length m on the grid. Let r be the weight of the upper right point of the rectangle and s be the weight of the lower left point where both belong to x. Then the upper left and lower right points r' and s' belong to y (See Figure 3). Then we have $s' = r+m$ and $r' = s-m$. So $I(y, X)-I(X, y) = I(x, X)-I(X, x)-r-s+r'+s' = I(x, X)-I(X, x)$. □

From these lemmas, it easily follows that for every grid diagram we must have $I(x, X) - I(X, x) = n$ and similarly $I(x, O) - I(O, x) = n$ if x is a generator in our chain complex. This is just an application of the well-known fact that every permutation can be written as a product of transpositions. This identity will be useful in simplifying the formulas for the Alexander and Maslov gradings and as well in our computation of the genus of torus knots.

3 Computing the genus

The genus of a p,q torus knot is

$$\frac{(p-1)(q-1)}{2}$$

It was also found in [OS04] that a knot's maximal Alexander grading(the highest Alexander grading which is nontrivial) is the genus. Using this result

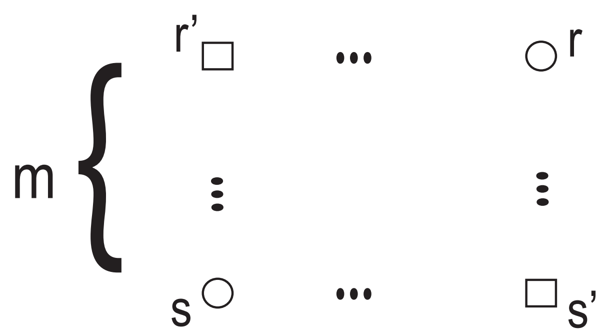

Figure 3: A detail for a grid diagram where the two generators x and y differ by a transposition

and by finding general equations for the maximal and minimal Alexander gradings of any p,q torus knot, we were able to confirm (1).

To find equations for the maximal and minimal Alexander gradings, it is necessary to see that all p,q torus knots can be represented on a grid diagram with grid number p+q [MB98], a top left to bottom right diagonal of X's or O's, and parallel diagonals of the alternate (O's or X's) terminating at the edges and of length p and q respectively. That this representation describes any p,q torus knot can be seen by reversing the connection of X's and O's so that there are no crossings on the grid diagram. The resulting diagram will intersect one edge of the diagram p times and the other q times, as seen in the center diagram in Figure 4.

The Alexander grading in a grid diagram with grid number n is defined as

$$\begin{aligned} A_i(x) &= J(x - \frac{1}{2}(X+O), X_i - O_i) - \binom{n_i - 1}{2} \\ &= J(x, X) - J(x, O) - \frac{1}{2}J(X,X) + \frac{1}{2}J(O,O) - \binom{n-1}{2} \end{aligned}$$

where for $A, B \subseteq \mathbb{R}^2$ $J(A,B) = \frac{I(A,B)+I(B,A)}{2}$ where $I(a,b)$ is the number of pairs $(a_1, a_2) \in A$ and $(b_1, b_2) \in B$ such that $a_1 < b_1$ and $a_2 < b_2$.

Only the first two terms of the equation are dependent on the specific matching, and need to be considered for maximizing or minimizing the

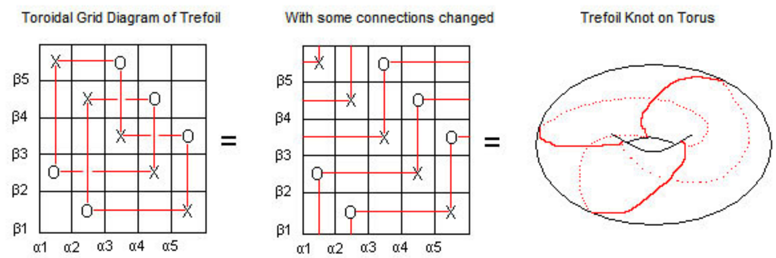

Figure 4: Representation of trefoil on toroidal grid diagram

Alexander gradings of a torus knot. These two terms can be simplified. By using the identities $I(x,X) - I(X,x) = n$ and $I(x,O) - I(O,x) = n$ one observes that minimizing or maximizing the Alexander gradings corresponds to minimizing or maximizing the following quantities:

$$\begin{aligned} A_{res}(x) &= I(x,X) - I(x,O) \\ \text{or } A_{res}(x) &= I(X,x) - I(O,x) \end{aligned}$$

It is not too difficult to see that the quantity $A_{res}(x)$ corresponds to the sum of winding numbers of each point on the matching with the knot oriented from O's to X's in each column. In the general grid diagram for a torus knot in Figure 5, $A_{res}(x) = 0$ for all matchings in areas a and d. This is seen from the previous equations and because any point in the region a will be above no X's or O's and in region d will be below no X's or O's. $A_{rel}(x)$ will be negative for matchings in areas b and c because there are no X's and always O's below any x in area b and in area c any x is always below an O but never below any X. Therefore the maximum value of $A_{rel}(x)$ is 0, and because of the structure of the grid diagram the unique matching where this is possible is forced to be the matching with points in the upper left corner of all O's. Since the grid diagram is a torus, it can be shifted upwards by a Dynnikov type 3 move, cyclic permutation, shown in Figure 6. Then by similar reasons, the values of $A_{rel}(x)$ around the center diagonal will be all positive, the values in the corners are all 0, forcing the unique matching with the least relative Alexander grading to be the matching with points in the upper left corner of all X's.

As all torus knots will have a grid diagram of the same general form, and the maximal and minimal Alexander grading can be computed in general terms for a $T_{p,q}$ torus knot in terms of p and q. Looking at each component

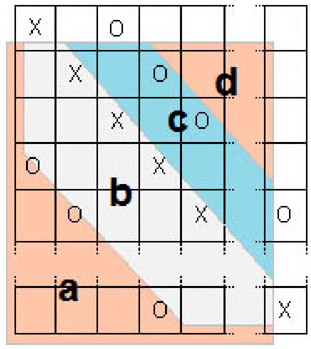

Figure 5: A general form of a grid diagram for a p,q torus knot

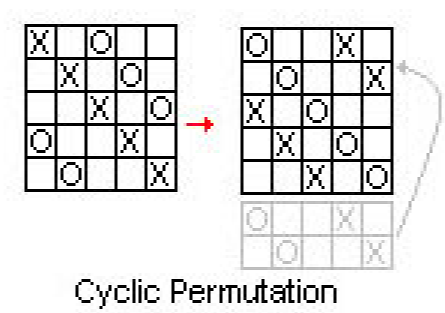

Figure 6: A Dynnikov type 3 shift

of A(x) in terms of p and q gives the following formulas

$$\begin{aligned} A_{min}(x) &= \frac{1}{2}\Big[(p+q)-(p+q)(p+1) \\ &\quad +2\sum_{i=1}^{min(p,q)} i+(p+q-1-2min(p,q))min(p,q) \\ &\quad +pq+p+q-1\Big] \\ \text{for } p>q,\ A_{max}(x) &= \frac{1}{2}\Big[p(q-1)+2\sum_{i=1}^{q} i+q(p-q)-q(p+1) \\ &\quad -p(q-1)+pq-(p+q-1)\Big] \\ \text{for } p<q,\ A_{max}(x) &= \frac{1}{2}\Big[p(q-1)+2\sum_{i=1}^{p} i+(q-p)(p+1)-q(p+1) \\ &\quad -p(q-1)+pq-(p+q-1)\Big] \end{aligned}$$

These simplify to

$$\begin{aligned} A_{min}(x) &= \frac{-pq-p-q+1}{2} \\ A_{max}(x) &= \frac{pq-p-q+1}{2} = \frac{(p-1)(q-1)}{2} = genus \end{aligned}$$

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