(1) With the definition of a vector bundle from class, show that the vector space operations define continuous maps

$$+ : E \times_B E \to E$$
$$\times : \mathbb{R} \times E \to E.$$

(Here, $E \times_B E$ is the fiber product $\{(e_1, e_2) \in E_1 \times E_2 \mid \pi(e_1) = \pi(e_2)\}$.)

(2) Suppose you are given the following data:
- Topological spaces $B$ and $F$.
- A set $E$ and a map of sets $\pi : E \to B$.
- An open cover $\mathcal{U} = \{U_i\}$ of $B$ and for each $i$ a bijection $\phi_i : \pi^{-1}(U_i) \to U_i \times F$ so that the following diagram (of sets) commutes:

$$\begin{array}{ccc}
\pi^{-1}(U_i) & \xrightarrow{\phi_i} & U_i \times F \\
\downarrow{\pi} & & \downarrow{\pi_1} \\
U_i & & 
\end{array}$$

Give conditions on the maps $\phi_i$ so that there is a topology on $E$ making $\pi : E \to B$ into a fiber bundle with $\{(U_i, \phi_i)\}$ an atlas.

(3) State and prove the analogue of the previous problem for vector bundles and for principal $G$-bundles.

(4) Write out the details of the proof that the bundle of frames of a vector bundles is a principal $GL(n)$-bundle.

(5) An oriented $n$-dimensional vector bundle is a vector bundle $\pi : E \to B$ together with an orientation of each fiber $E_b$, so that these orientations are continuous in the following sense. For each $b \in B$ there is a chart $(U \ni b, \phi : \pi^{-1}(U) \to U \times \mathbb{R}^n)$ so that for all $b' \in U$, $\phi|_{E_{b'}} : E_{b'} \to \mathbb{R}^n$ is orientation-preserving. Show that given an oriented $n$-dimensional vector bundle there is an induced principal $GL_+(\mathbb{R}^n)$-bundle (the “bundle of oriented frames”), and conversely given a principal $GL_+(\mathbb{R}^n)$-bundle there is an induced oriented $n$-plane bundle.

(6) A Riemannian metric on a vector bundle $\pi : E \to B$ is an inner product $\langle \cdot, \cdot \rangle_b$ on each fiber $E_b$ of $E$, which is continuous in the sense that the induced map $E \oplus E = E \times_B E \to \mathbb{R}$ is continuous. Show that given a Riemannian metric on a vector bundle $E$, there is an induced principal $O(n)$-bundle (the “bundle of orthonormal frames”), and conversely given a principal $O(n)$-bundle there is an induced vector bundle with Riemannian metric.

(7) What operation on principal $O(n)$-bundles corresponds to dualizing a vector bundle? The direct sum of vector bundles?
(8) For nice spaces \( X \) (e.g., CW complexes) and abelian groups \( G \), there is a canonical isomorphism \( \hat{H}^i(X; G) \cong H^i(X; G) \) between the Čech and singular cohomology of \( X \) with coefficients in \( G \). A nice, readable proof can be found in Frank Warner’s *Foundations of Differential Manifolds and Lie Groups*, Chapter 5. In the rest of this problem, cohomology either means Čech cohomology or singular cohomology after applying this isomorphism.

(a) Let \( \pi: E \to B \) be an \( n \)-dimensional vector bundle or, equivalently, a principal \( GL(n, \mathbb{R}) \)-bundle, given by a Čech cocycle \( \phi \in \hat{H}^1(B; GL(n, \mathbb{R})) \). Show that the sign of the determinant \( \text{sign} \circ \det: GL(n, \mathbb{R}) \to \{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z} \) induces a map

\[
\hat{H}^1(B; GL(n, \mathbb{R})) \to \hat{H}^1(B; \mathbb{Z}/2\mathbb{Z})
\]

and so \( \phi \) induces an element \( w_1(E) \in H^1(B; \mathbb{Z}/2\mathbb{Z}) \).

(b) Compute \( w_1 \) for the trivial line bundle (1-dimensional vector bundle) over the circle and for the Möbius band.

(c) Prove that (for nice spaces) a line bundle \( \pi: E \to B \) is trivial if and only if \( w_1(E) = 0 \in H^1(B; \mathbb{Z}/2\mathbb{Z}) \).

(9) Show that the exact sequence of abelian topological groups

\[
0 \to \mathbb{Z} \to \mathbb{R} \to S^1 = GL(1, \mathbb{C}) \to 0
\]

induces an exact sequence in Čech cohomology

\[
\hat{H}^1(B; \mathbb{Z}) \to \hat{H}^1(B; \mathbb{R}) \to \hat{H}^1(B; S^1) \overset{\delta}{\to} \hat{H}^2(B; \mathbb{Z}).
\]

Given a complex line bundle (principal \( GL(1, \mathbb{C}) \)-bundle) \( \pi: E \to B \) coming from cocycle data \( \phi \in \hat{H}^1(B; GL(1, \mathbb{C})) \), let \( c_1(E) = \delta(\phi) \). Compute \( c_1(E) \) for some complex line bundles over \( S^2 \).

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