# MATH 690 HOMEWORK 2
# OCTOBER 2, 2017.

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(1) Let $E$ be a fiber bundle over $B$ with fiber $F$ and structure group $G$. Let

$$P = \bigcup_{x \in B} \{G \text{ homeos } E_x \to F\}.$$

Give a topology and $G$-action on $P$ so that the obvious projection $P \to B$ makes $P$ into a principal $G$-bundle. Prove that your $P$ is a principal $G$-bundle. Show that $P$ is equivalent to $E$ in the sense that they are specified by the same Čech data.

(2) In class, we defined a map $f\colon P \to EG$ from a section $s$ of $P \times_G EG$ by declaring $f(p) = x$ where $s(\pi(p)) = [p, x]$. Show that $f$ is continuous. (Hint: use a chart for $P$.)

(3) The *Spin* group:
   (a) Show that for each $n \geq 2$, the group $SO(n)$ has a unique connected 2-fold covering space. Call this space $\mathit{Spin}(n)$.
   (b) Use the group structure of $SO(n)$ to induce a group structure on $\mathit{Spin}(n)$.
   (c) What familiar space is $\mathit{Spin}(3)$? Also, show that as topological groups, $\mathit{Spin}(3) \cong SU(2)$.

(4) For non-injective (or non-effective) structure groups $G$, we imposed an extra requirement that the lifts $\widetilde{\phi}_{ij} \to G$ of the transition maps $\phi_{ij}\colon U_i \cap U_j \to \mathit{Homeo}(F, F)$ form a cocycle. Give an example of an oriented $n$-plane bundle with transition maps $\phi_{ij}$ which do lift to $\mathit{Spin}(n)$ but where the lifts cannot be chosen to satisfy the cocycle condition. (Hint: decompose the sphere as a union of two hemispheres and consider the Čech cocycle induced by a generator of $\pi_1(SO(3)) = \mathbb{Z}/2$, say. This corresponds to an oriented 3-plane bundle over $S^2$, and does not lift to a *Spin*-bundle. Further divide the sphere into open sets so that the intersection of each pair is contractible. Observe that you can lift the cocycle on each of these intersections, but that the lifts do not form a cocycle.)

(5) More on existence of *Spin*-structures:
   (a) Show that there is an exact sequence of Čech cohomology groups

$$\check{H}^1(X; \mathbb{Z}/2) \to \check{H}^1(X; \mathit{Spin}(n)) \to \check{H}^1(X; SO(n)) \xrightarrow{\delta} \check{H}^2(X; \mathbb{Z}/2)$$

   (b) Given an $SO(n)$-bundle $P$ coming from $\alpha \in \check{H}^1(X; SO(n))$, let $w_2(P) = \delta(\alpha)$. Deduce that $P$ admits a lift to a $\mathit{Spin}(n)$-bundle if and only if $w_2(P) = 0$.
   (c) Compute $w_2$ for your example in the previous problem.
   (d) Extend the exact sequence from part (5a) to an exact sequence

$$0 \to \check{H}^1(X; \mathbb{Z}/2) \to \check{H}^1(X; \mathit{Spin}(n)) \to \check{H}^1(X; SO(n)) \to \check{H}^2(X; \mathbb{Z}/2)$$

   and deduce that if an $SO(n)$-bundle $E$ admits a $\mathit{Spin}(n)$-lift then the set of lifts form an affine space for $H^1(X; \mathbb{Z}/2)$.

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