MATH 690 HOMEWORK 3
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(1) With notation as in class, show that G acts on $\mathscr{N}\mathscr{E}G$ with quotient $\mathscr{N}\mathscr{B}G$.

(2) Suppose that $\mathscr{C}$ is a category with an initial (or terminal) object. Show that the homology groups of the nerve $\mathscr{N}\mathscr{C}$ vanish, and $\pi_1\mathscr{N}\mathscr{C} = 0$, giving another proof that $\mathscr{N}\mathscr{C}$ is contractible.

(3) (Segal, Proposition 2.1.)
   (a) Suppose that $F\colon \mathscr{C} \to \mathscr{D}$ is a functor. Show that there is an induced continuous map $\mathscr{N}F\colon \mathscr{N}\mathscr{C} \to \mathscr{N}\mathscr{D}$ of nerves.
   (b) Suppose that $F_0, F_1\colon \mathscr{C} \to \mathscr{D}$ are functors and η is a natural transformation from F_0 to F_1. Show that $\mathscr{N}F_0$ and $\mathscr{N}F_1$ are homotopic. (Hint: think of η as a map from $\mathscr{C} \times (\bullet \to \bullet)$ to $\mathscr{D}$.)
   (c) Give yet another proof that if $\mathscr{C}$ has an initial (or terminal) object then $\mathscr{N}\mathscr{C}$ is contractible.

(4) Prove that the canonical vector bundle γ_n over $\mathrm{Gr}(n, \mathbb{R}^K)$ is, in fact, a vector bundle.

(5) In class, we gave two definitions of the Bockstein homomorphism:
   (a) Via the connecting homomorphism associated to the coefficient sequence
$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$$
   (b) As the element induced by the generator of
$$\mathbb{Z}/2 = \mathrm{Ext}^1(\mathbb{Z}/2, \mathbb{Z}/2) = \mathrm{Ext}^1(H_n(K(\mathbb{Z}/2, n)), \mathbb{Z}/2) \subset H^{n+1}(K(\mathbb{Z}/2, n); \mathbb{Z}/2)$$
   via the Yoneda lemma.

   Prove that these two definitions agree.

(6) *Spin* structures again.
   (a) Let $f\colon G \to H$ be a continuous homomorphism of (reasonable) topological groups. Show that for appropriate choices of BG and BH there is an induced fibration $BG \to BH$. (Hint: choose any EG and EH and observe that G acts on $EG \times EH$, using f.)
   (b) Setting $G = \mathit{Spin}(n)$ and $H = SO(n)$, what is the fiber of the fibration $B\,\mathit{Spin}(n) \to B\,SO(n)$?
   (c) Show that this fibration is principal (see the section in Hatcher on Postnikov towers).
   (d) Give another construction of the obstruction w_2 to existence of a *Spin*-structure and another proof that the set of *Spin*-structures on an oriented vector bundle E with $w_2(E) = 0$ is in bijection with $H^1(B; \mathbb{Z}/2)$.

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