MATH 690 HOMEWORK 4
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(1) Recall that an *orientation* of an n-dimensional disk bundle $D \to B$ is a choice of generator o_b of $H_n(D_b, \partial D_b) \cong \mathbb{Z}$ for each $b \in B$, so that for each chart $(U, \phi\colon \pi^{-1}(U) \to U \times D^n)$ for B and all $b, b' \in U$, the compositions

$$H_n(D_b, \partial D_b) \to H_n(U \times D^n, U \times S^{n-1}) \to H_n(D^n, S^{n-1}) \qquad \text{and}$$
$$H_n(D_{b'}, \partial D_{b'}) \to H_n(U \times D^n, U \times S^{n-1}) \to H_n(D^n, S^{n-1})$$

send o_b and $o_{b'}$ to the same element.

Show that if E is an oriented vector bundle with Riemannian metric then the disk bundle DE inherits an orientation.

(2) Deduce the Gysin sequence and Thom isomorphism theorem from the Serre spectral sequence.

(3) Use the Gysin sequence to compute $H^*(\mathbb{C}P^n)$, via the fiber bundle $S^1 \to S^{2n-1} \to \mathbb{C}P^n$.

(4) Compute the Euler class of the canonical line bundle over $\mathbb{C}P^n$, directly from its definition.

(5) Compute the Euler class of the tangent bundle to S^2, directly from its definition.

(6) Given an n-dimensional disk bundle DE over a CW complex X and a section s of $SE|_{X^{n-1}}$ over the $(n-1)$-skeleton, we associated a cellular cochain $e_s \in C^n_{cell}(X)$.
  (a) Show that if s' is another section of $SE|_{X^{n-1}}$ then $e_s - e_{s'}$ is a coboundary.
  (b) Show that for any $\alpha \in C^{n-1}_{cell}(X)$ there is another section s' so that $e_s - e_{s'} = \delta(\alpha)$.

(7) Let X be a finite CW complex. Let $C(X)$ be the ring of continuous maps $X \to \mathbb{R}$, with pointwise addition and multiplication. Given a vector bundle $E \to X$, the set of sections of E is a $C(X)$-module. Show that the set of sections is a projective $C(X)$-module and, conversely, every finitely-generated projective $C(X)$-module arises this way. (Hint: use the fact that E is a subbundle of a trivial bundle.)

(8)

(9) Given a closed, oriented, smooth submanifold N^n of a closed, oriented, smooth manifold M^m, we have two classes in $H^*(M)$:
  (a) The image of the Thom class $u_{\nu N} \in H^{m-n}(D\nu N, S\nu N) \cong H^{m-n}(M, M \setminus N)$ under the restriction map $H^{m-n}(M, M \setminus N) \to H^{n-m}(M)$.
  (b) The Poincaré dual of the fundamental class of N, $PD[N] \in H^{m-n}(M)$.

Prove that these classes are equal.

(10) Let M be an oriented, smooth manifold, $f\colon M \to \mathbb{R}$ a Morse function and p a critical point of f. Let Z be the zero-section of TM and $\Gamma_{\nabla f}$ the graph of the gradient of f. Show that the sign of the intersection point $p \in Z \cap \Gamma_{\nabla f}$ is $(-1)^{\mathrm{ind}(p)}$.

(11) Milnor-Stasheff, Problem 11-C.

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