MATH 690 HOMEWORK 7
NOVEMBER 27, 2017.

ROBERT LIPSHITZ

(1) Basics of bordism...
  (a) Recall that for closed, smooth n-manifolds M_1 and M_2 we declared that $M_1 \sim M_2$ if there is a compact, smooth $(n+1)$-manifold-with-boundary N so that $\partial N = M_1 \amalg M_2$. Prove that $\sim$ is an equivalence relation. (Hint: transitivity does need at least a little work.)
  (b) Let $\mathfrak{N}$ denote the disjoint union over n of the set of $\sim$-equivalence classes of n-manifolds. Show that $\amalg$ and $\times$ make $\mathfrak{N}$ into a commutative ring.

(2) Give formal definitions of the complex bordism groups and the spin bordism groups. (Recall that you should use structures on the stable normal bundles.) Compute the first couple of complex bordism and spin bordism groups.

(3) Show that if a closed manifold M is the boundary of a compact $(n+1)$-manifold-with-boundary then all Stiefel-Whitney numbers of M vanish.

(4) Verify that the signature satisfies $\sigma(M_1 \amalg M_2) = \sigma(M_1) + \sigma(M_2)$ and $\sigma(M_1 \times M_2) = \sigma(M_1)\sigma(M_2)$. What if M_1 and M_2 are *not* $4n$-dimensional?

(5) Let V be a vector space over $\mathbb{R}$ and $B \colon V \otimes V \to \mathbb{R}$ a bilinear form. suppose that there is a half-dimensional subspace L of V so that $B(v, w) = 0$ for all $v, w \in L$. Prove that the signature of B is zero.

(6) Fix a closed, smooth, oriented M^7 with $H^3(M) = H^4(M) = 0$. Let B_1 and B_2 be oriented 8-manifolds-with-boundary with $\partial B_j = M$. Let $C = B_1 \cup_M (-B_2)$. Let $i \colon H^4(B_j, M) \to H^4(B_j)$ be the map from the long exact sequence of a pair. Define $\sigma(B_j)$ to be the signature of the bilinear form

$$\begin{aligned} H^4(B_j, M; \mathbb{Q}) \otimes H^4(B_j, M; \mathbb{Q}) &\to \mathbb{Q} \\ (a, b) &\mapsto \langle a \cup b, [B_j] \rangle. \end{aligned}$$

Prove Milnor's lemma that

$$\begin{aligned} \langle p_1(TC)^2, [C] \rangle &= \langle i^{-1} p_1(TB_1)^2, [B_1] \rangle - \langle i^{-1} p_1(TB_2)^2, [B_2] \rangle \\ \sigma(C) &= \sigma(B_1) - \sigma(B_2). \end{aligned}$$

(7) Prove that if a smooth, closed n-manifold M admits a Morse function with two critical points then M is homeomorphic to S^n.

(8) Milnor-Stasheff, Problem 16-C.

(9) Milnor-Stasheff, Problem 16-D.

(10) Milnor-Stasheff, Problem 16-E.

(11) Milnor-Stasheff, Problem 16-F.

E-mail address: lipshitz@uoregon.edu