MATH 634 HOMEWORK 4
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INSTRUCTOR: ROBERT LIPSHITZ

Updated 10/15: typos in problem 7 corrected.

(1) Hatcher 1.1.2 (p. 38).
(2) Let G be a topological group and $e \in G$ the identity element. Show that $\pi_1(G, e)$ is abelian. (Hint: there are two ways you can multiply loops in G.)
(3) Hatcher 1.1.5 (p. 38). Be careful about basepoints where appropriate.
(4) Hatcher 1.1.7 (p. 38).
(5) Hatcher 1.1.9 (p. 38). (This is tricky, so some hints are on the next page if you want them. You may assume any reasonable facts you want about how measures of sets work.)
(6) Hatcher 1.1.18 (p. 39).
(7) In class, we defined π_1 by using maps

$$\Delta\colon S^1 \to S^1 \vee S^1 \qquad r\colon S^1 \to S^1 \qquad \ell\colon S^1 \vee S^1 \to S^1.$$

We asserted that

$$\ell \circ (\mathbb{I} \vee r) \circ \Delta \sim \iota \circ \epsilon \sim \ell \circ (r \vee \mathbb{I}) \circ \Delta.$$

Prove this.
(8) Using the previous problem, verify that every element of $\pi_1(X, x_0)$ has an inverse.
(9) Let $H^1(X) = [X, S^1]$ denote the set of homotopy classes of continuous maps from X to S^1. (There are no basepoints in this discussion.)
   (a) Recall that S^1 is a topological group. Use the group structure on S^1 to make $H^1(X)$ into a group. Note that this group is abelian for any X.
   (b) Compute $H^1(\{pt\})$. (This should be easy.)
   (c) Compute $H^1(S^1)$. (Use the fact that $\pi_1(S^1) \cong \mathbb{Z}$.)
   (d) Show that H^1 is *functorial* in the following sense: if $f\colon X \to Y$ is continuous then there is an induced homomorphism $f^*\colon H^1(Y) \to H^1(X)$. Moreover, if $g\colon Y \to Z$ then $(g \circ f)^* = f^* \circ g^*\colon H^1(Z) \to H^1(X)$. (This should be easy.)
   (e) Show that if $f \sim g$ then $f^* = g^*$. Conclude that if $X \simeq Y$ then $H^1(X) \cong H^1(Y)$. (This should be easy.)
   (f) Use H^1 to prove that there is no retraction $\mathbb{D}^2 \to S^1$, the key step in proving the Brouwer fixed point theorem.

To think about but not turn in:

(1) Problems 1.1.1, 1.1.3, and 1.1.16 seem like particularly good practice.

(2) Read through the remaining problems in Section 1.1, and do any that seem difficult or surprising.

Hints for Hatcher 1.1.9:

(1) Given a vector $v \in S^2$ and a real number λ, there is a corresponding plane $P_{v,\lambda} = \{w \in \mathbb{R}^3 \mid w \cdot v = \lambda\}$. Notice that $P_{v,\lambda} = P_{-v,-\lambda}$.
(2) The proof is easier if you assume that for each v there is exactly one λ so that $P_{v,\lambda}$ cuts A_1 into two equal pieces. Solve this case first.

E-mail address: `lipshitz@uoregon.edu`