MATH 431/531 FALL 2021
HOMEWORK 1
DUE OCTOBER 1, 2021.

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Note. This homework assignment is intended partly to make sure you are ready for the course. So, it is about a third the length of, and somewhat easier than, a typical homework assignment.

Math 531 students. This week only it is okay to turn in your homework neatly written, instead of typed in LaTeX.

Problems:

(1) Let X be a set, and d_{dis} be the *discrete metric* on X, defined by

$$d_{dis}(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y. \end{cases}$$

Prove that (X, d_{dis}) is a metric space.

(2) Consider $\mathbb{R}^2$ with the standard metric $d((x,y),(x',y')) = \sqrt{(x-x')^2 + (y-y')^2}$. Prove that $U = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 4 \text{ and } y > 0\}$ is open in $\mathbb{R}^2$.

(3) Let X and Y be sets. Show that there is a bijection between the Cartesian product $X \times Y$ and the set of functions

$$\{f \colon \{0,1\} \to X \cup Y \mid f(0) \in X \text{ and } f(1) \in Y\}.$$

Challenge problems (required for Math 531, optional for 431):

(4) The 2-*adic metric* on the set of rationals $\mathbb{Q}$ is defined as follows. Given $n \in \mathbb{Z}$, define $\operatorname{ord}_2(n) = \max\{r \in \mathbb{N} \mid 2^r \text{ divides } n\}$ if $n \neq 0$. For $p/q \in \mathbb{Q}$, define $\operatorname{ord}_2(p/q) = \operatorname{ord}_2(p) - \operatorname{ord}_2(q)$. Then we define

$$d(p/q, r/s) = \begin{cases} 2^{-\operatorname{ord}_2(p/q - r/s)} & \text{if } p/q \neq r/s \\ 0 & \text{if } p/q = r/s. \end{cases}$$

(a) Compute $d(1,2)$, $d(1,3)$, $d(1,5)$, $d(1,1/2)$, $d(1/2,1/3)$.

(b) Prove that $(\mathbb{Q}, d)$ is a metric space.

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