Problems:

(1) Prove parts (b,d,g) of Proposition 2.8 in Lee.
(2) Let $X$ be a topological space and $A$ a subset of $X$. There is a map $i: A \to X$ given by $i(a) = a$; $i$ is called the inclusion map.
   (a) Prove that if we give $A$ the subspace topology then the inclusion map is continuous.
   (b) Given topologies $\mathcal{U}$, $\mathcal{V}$ on $A$, we say that $\mathcal{V}$ is finer than $\mathcal{U}$ if $\mathcal{U} \subset \mathcal{V}$, i.e., if every set which is open with respect to $\mathcal{U}$ is also open with respect to $\mathcal{V}$.
   Suppose that $\mathcal{V}$ is a topology on $A$ so that the inclusion map is continuous with respect to $\mathcal{V}$. Prove that $\mathcal{V}$ is finer than the subspace topology. (In other words, the subspace topology is the coarsest topology for which the inclusion map is continuous.)
(3) Let $X$ be a set. What subsets of $X$ are dense in $X$ with respect to the discrete topology? With respect to the indiscrete topology?
(4) Let $A = (−7, 0) \cup \{1/n \mid n \in \mathbb{Z}_{>0}\} \subset \mathbb{R}$, with the usual (metric) topology. (Here, $\mathbb{Z}_{>0}$ denotes the positive integers. Find:
   (a) The interior of $A$.
   (b) The exterior of $A$.
   (c) The closure of $A$.
   (d) The boundary of $A$.
   (e) The limit points of $A$.
   (f) The isolated points of $A$.
   No justification needed in this problem; just the answers.
(5) Consider the unit circle $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ as a subspace of $\mathbb{R}^2$. What is the boundary of the subspace $A$? (Justification optional.)
(6) Find a function $f: \mathbb{R} \to \mathbb{R}$ (both with the standard topology) which is
   (a) Discontinuous at every point.
   (b) Continuous at exactly one point.
   (c) Continuous exactly at $\mathbb{Z} \subset \mathbb{R}$.
   (Here, we are using the metric space definition of continuity at a point.)

Challenge problems (required for Math 531, optional for 431):

(7) (a) Solve Lee Exercise 2.6.
   (b) Using the previous part, show that if $X$ is a topological space and $A$ is a subspace of $X$ then there is a unique coarsest topology on $A$ so that the inclusion map $i: A \to X$ is continuous. Do not use the explicit description of the subspace topology.
Consider \((0,1)^2\) with the lexicographic order, i.e.,

\[
(a,b) < (c,d) \equiv \begin{cases} 
   a < c & \text{or} \\
   a = c \; \text{and} \; b < d
\end{cases}
\]

Let \(U\) be the order topology on \((0,1)^2\) corresponding to this order \(<\). Find a metric on \((0,1)^2\) so that the induced topology is \(U\). (i.e., show that this topology is \textit{metrizable}).

**Bonus problems** (not required for anyone):

9) Consider \([0,1]^2\) with the lexicographic order. Show that the order topology on \([0,1]^2\) is \textit{not} metrizable (i.e., is not induced by any metric).

10) Let \(X\) be a topological space and \(A\) a subset of \(X\). The \textit{Cantor derivative} of \(A\) is the set of limit points of \(A\).
   
   (a) Find a nonempty set \(A \subset \mathbb{R}\) whose first Cantor derivative is empty.
   
   (b) Find a set \(A \subset \mathbb{R}\) whose first Cantor derivative is non-empty but whose second Cantor derivative is empty.
   
   (c) For any positive integer \(n\), find a set \(A \subset \mathbb{R}\) whose \((n-1)^{\text{st}}\) Cantor derivative is non-empty but whose \(n^{\text{th}}\) Cantor derivative is empty.
   
   (d) The \(\omega^{\text{th}}\) Cantor derivative of \(A\) is

   \[
   A^{(\omega)} = A \cap A' \cap A'' \cap A''' \cap \ldots
   \]

   (where I am using primes to denote Cantor derivatives). Find a subset \(A \subset \mathbb{R}\) so that all finite Cantor derivatives of \(A\) are non-empty but \(A^{(\omega)}\) is empty.

   (e) Given any countable ordinal \(\alpha\) define the \(\alpha^{\text{th}}\) Cantor derivative of \(A\). (Hint: there are two cases, depending on whether \(\alpha\) is a successor ordinal or a limit ordinal.)
   
   Find a subset \(A \subset \mathbb{R}\) so that if \(\beta < \alpha\) then \(A^{(\beta)} \neq \emptyset\) but \(A^{(\alpha)} = \emptyset\).

   (f) A closed subset \(A \subset \mathbb{R}\) is \textit{perfect} if \(A = A'\). Show that for any closed subset \(A \subset \mathbb{R}\) there is a countable ordinal \(\alpha\) so that \(A^{(\alpha)}\) is perfect.

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