MATH 636 HOMEWORK 1
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Problems to turn in:

(1) Hatcher 3.3.24 (p. 259). Deduce also that if M is simply connected then $H_i(M) \cong H_i(S^3)$ for each i.

(2) Hatcher 3.3.34 (p. 260).

(3) Let X be a closed, orientable $4n$-dimensional manifold.

  (a) Recall that the *signature* of a bilinear form is the number of positive eigenvalues minus the number of negative eigenvalues. Prove the following algebraic lemma.
  Lemma. Let V be a $2k$-dimensional $\mathbb{R}$-vector space and $g\colon V \times V \to \mathbb{R}$ a nondegenerate, symmetric, bilinear form. If there is a k-dimensional subspace $W \subset V$ so that $g(v, w) = 0$ for all $v, w \in W$ then the signature of g is zero.

  (b) Fix an orientation $[X]$ for X. Define the *signature* $\sigma(X)$ of X to be the signature of the bilinear form
$$\begin{aligned} H^{2k}(X;\mathbb{R}) \times H^{2k}(X;\mathbb{R}) &\to \mathbb{R} \\ (a,b) &\mapsto \langle a \cup b, [X]\rangle. \end{aligned}$$
  Show that if X is the boundary of a compact, orientable $(4n+1)$-dimensional manifold-with-boundary then $\sigma(X) = 0$.

  (c) Deduce that $\mathbb{C}P^{2n}$ is not the boundary of an orientable $(4n+1)$-dimensional manifold.

Review / qualifying exam practice (not to turn in):

(1) Hatcher 3.3.33.

(2) Use Lefschetz duality to show that the cup product
$$H^k(D^k \times D^\ell, S^{k-1} \times D^\ell) \otimes H^\ell(D^k \times D^\ell, D^k \times S^{\ell-1}) \xrightarrow{\cup} H^{k+\ell}(D^k \times D^\ell, \partial(D^k \times D^\ell))$$
is an isomorphism.

(3) A *2-component link* is a pair of disjoint, smooth embeddings $K_1, K_2\colon S^1 \hookrightarrow S^3$. Given a 2-component link, the *linking number* of L is defined as follows:

  (a) By Alexander duality, $H_1(S^3 \setminus K_1(S^1)) \cong H^1(K_1(S^1)) \cong H^1(S^1) \cong \mathbb{Z}$, where the last isomorphism sends the positive generator of $H^1(K_1) = H^1(S^1)$ to 1. On the other hand, $(K_2)_*([S^1])$ is an element of $H_1(S^3 \setminus K_1(S^1))$. The image of this element in $\mathbb{Z}$ is the linking number of the link.

  (b) Alternatively, let $\mathrm{nbd}(K_1(S^1))$ be a small solid torus neighborhood of $K_1(S^1)$. Then $H_2(S^3 \setminus \mathrm{nbd}(K_1(S^1)), \partial\,\mathrm{nbd}(K_1(S^1))) \to H_1(\partial\,\mathrm{nbd}(K_1(S^1))) \to H_1(K_1(S^1))$ is an isomorphism (this needs proof), so the generator $(K_1)_*([S^1])$ gives a preferred generator of $H_2(S^3 \setminus \mathrm{nbd}(K_1(S^1)), \partial\,\mathrm{nbd}(K_1(S^1)))$. By Lefschetz duality, $H_2(S^3 \setminus \mathrm{nbd}(K_1(S^1)), \partial\,\mathrm{nbd}(K_1(S^1))) \cong H^1(S^3 \setminus \mathrm{nbd}(K_1))$. Evaluating this preferred generator on $(K_2)_*[S^1]$ gives an integer, the linking number of the link.

Show that these two definitions are equivalent.

More problems to think about but not turn in:

(1) Let Y be a compact, orientable 3-manifold-with-boundary, and $F = \partial Y$, a closed surface of genus g.
   (a) Fix an orientation (fundamental class) $[F]$ for F. Define a pairing $\omega\colon H^1(F) \times H^1(F) \to \mathbb{Z}$ by
   $$\omega(a, b) = \langle a \cup b, [F]\rangle = (a \cup b) \cap [F].$$
   Prove that ω is skew-symmetric and non-degenerate.
   (b) Let $K \subset H^1(F)$ be the image of $j^*\colon H^1(Y) \to H^1(F)$ (where $j\colon F \to Y$ is inclusion). Show that for any $a, b \in K$, $\omega(a, b) = 0$.
   (c) Deduce from Hatcher, problem 3.3.28, that the rank of K is at most g. In particular, deduce that if $g > 0$ then Y is not simply connected.
   (d) Show that the rank of K is exactly g. (In other words, K is a Lagrangian subspace of $(H^1(F), \omega)$.)

(2) Hatcher 3.3.35.

(3) Read through Hatcher's first computation of the ring structure on $H^*(\mathbb{C}P^n)$ with Poincaré duality in mind, and see how he came up with it.

(4) Let Y be a closed, smooth, oriented 3-manifold. Show that every homology class $\alpha \in H_2(Y; \mathbb{Z})$ is represented by some embedded surface $\Sigma \subset Y$, as follows. Let $b = \mathrm{PD}(\alpha) \in H^1(Y; \mathbb{Z})$. From last quarter, there is a map $f\colon Y \to S^1$ so that $f^*(d\theta) = b$, where $d\theta \in H^1(S^1; \mathbb{Z})$ is the preferred generator. (Here, perhaps we are using the fact that Y is homeomorphic to a CW complex.) Show that f is homotopic to a smooth map $g\colon Y \to S^1$. Let $p \in S^1$ be a regular value of g. Show that $\Sigma = g^{-1}(p)$ represents α.

(5) Let $p(z_0, z_1, z_2), q(z_0, z_1, z_2) \in \mathbb{C}[z_0, z_1, z_2]$ be degree d homogeneous polynomials so that, viewing p and q as maps $\mathbb{R}^6 \cong \mathbb{C}^3 \to \mathbb{C} \cong \mathbb{R}^2$, 0 is a regular value of p and q.
   (a) Show that $\{[z_0 : z_1 : z_2] \in \mathbb{C}P^2 \mid p(z_0, z_1, z_2)\}$ is a smooth surface Σ_p.
   (b) Show that there is a smooth path p_t of homogeneous polynomials so that $p_0 = p$ and $p_1 = q$, and 0 is a regular value of p_t for all t. (Hint: show that 0 not being a regular value is a polynomial equation on the coefficients of p_t, and hence has real codimension 2.)
   (c) Show that Σ_p and Σ_q represent the same element of $H_2(\mathbb{C}P^2)$, as follows. Let p_t be as in the previous part. Then
   $$\{(t, [z_0 : z_1 : z_2]) \in [0, 1] \times \mathbb{C}P^2 \mid p_t(z_0, z_1, z_2) = 0\}$$
   is an orientable 3-manifold inside $[0, 1] \times \mathbb{C}P^2$, with boundary $-\Sigma_p \coprod \Sigma_q$ (where the $-$ denotes orientation reversal). Hence, from an exercise last week, $[\Sigma_p] = [\Sigma_q] \in H_2([0, 1] \times \mathbb{C}P^2) = H_2(\mathbb{C}P^2)$.

(6) Given $n \in \mathbb{N}$, let $Y_n = \{z \in \mathbb{C} \mid z^n \in \mathbb{R}_{\geq 0}\}$, so Y_n looks like a graph with one central vertex and n spokes. A *branched surface* is a topological space X so that each point $x \in X$ has a neighborhood $U \ni x$ homeomorphic to either $\mathbb{R}^2$ or $(0, 1) \times Y_n$ for some n. (That is, near each point Y either looks like $\mathbb{R}^2$ or like n half-planes coming together.) A *p-branched surface* is a branched surface where the only Y_ns which occur have n divisible by p. (So, for instance, a 2-branched surface can look locally like a plane, like 4 half-planes coming together, like 6 half-planes coming together, and so on.)

(a) Suppose X is a compact 2-branched surface. Show that X has a fundamental class $[X] \in H_2(X; \mathbb{F}_2)$.
(b)
(c) (Harder) Read what the Bockstein homomorphism is and relate it to representing mod-2 homology classes by 2-branched surfaces.
(d) (Harder) Generalize the previous two parts to p-branched surfaces for p an odd prime. Here, you will need a notion of orientability for p-branched surfaces.

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