MATH 636 HOMEWORK 10
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Problems to turn in:

(P-1) Fix $n \geq 4$. Compute $H^{n+1}(K(\mathbb{Z},n);\mathbb{Z})$, $H^{n+2}(K(\mathbb{Z},n);\mathbb{Z})$, $H^{n+1}(K(\mathbb{Z},n);\mathbb{Z}/2)$, and $H^{n+2}(K(\mathbb{Z},n);\mathbb{Z}/2)$. (Hint: you know the $(n+1)$-skeleton of $K(\mathbb{Z},n)$. In class, we computed $\pi_{n+1}(S^n)$; use this to describe the $(n+2)$-skeleton. Then use the long exact sequence for a pair to figure out enough about π_{n+2} of the result to figure out enough about the $(n+3)$-skeleton to compute H^{n+2}.)

(P-2) Fix $n \geq 4$. Determine how many homotopy types of CW complexes X there are with

$$\pi_i(X) = \begin{cases} \mathbb{Z} & i = n \\ \mathbb{Z} & i = n+1 \\ 0 & \text{otherwise.} \end{cases}$$

Do the same for complexes Y with

$$\pi_i(Y) = \begin{cases} \mathbb{Z} & i = n \\ \mathbb{Z}/2 & i = n+1 \\ 0 & \text{otherwise.} \end{cases}$$

(Hint: use k-invariants and your solution to the previous problem.)

(P-3) Consider the question of whether an element $a \in H^n(X;\mathbb{Z}/p)$ comes from an element of $H^n(X;\mathbb{Z})$ via the map $H^n(X;\mathbb{Z}) \to H^n(X;\mathbb{Z}/p)$ induced by the usual map $\mathbb{Z} \to \mathbb{Z}/p$. Formulate this as an obstruction theory problem, and show that the obstruction is an element $\beta(a) \in H^{n+1}(X;\mathbb{Z})$. (This is an example of the Bockstein homomorphisms; see Hatcher for a simpler definition.)

(P-4) Hatcher 4.3.22 (p. 420). (You may assume E, B, C, and the spaces E', B' involved in the definition of a principal fibration are all CW complexes.)

(P-5) Hatcher 4.D.2 (p. 447).

(P-6) Hatcher 4.D.3 (p. 447). Just solve the complex case—skip the quaternionic one.

Review / qualifying exam practice (not to turn in):

(Q-1) Hatcher 4.3.17, 4.3.18, 4.3.20, 4.3.21.

(Q-2) Hatcher 4.D.1, 4.D.5.

(Q-3) Turn problem (P-1) around to show that if we know that $H^{n+2}(K(\mathbb{Z},n))$ is nontrivial then it follows that $\pi_{n+1}(S^n)$ $(n \geq 3)$ is also nontrivial. From there, it's not hard to show that $\pi_{n+1}(S^n) \cong \mathbb{Z}/2$ for $n \geq 3$ (how?).

(Q-4) Find examples illustrating that the hypotheses in the Leray-Hirsch theorem are necessary.

More problems to think about but not turn in:

(OP-1) Let E be a rank k complex vector bundle over X. Show that there is a well-defined primary obstruction $c_i \in H^{2i}(X)$ to finding $k-i+1$ linearly independent sections of E. (The c_i are the *Chern classes* of E.)

(OP-2) Let E be a rank k real vector bundle over X. Consider the problem of finding $k-i+1$ linearly independent sections of E. The condition that π_1 acts trivially on π_n required for obstruction theory to work may not be satisfied (in fact, it isn't in general). If it were satisfied, where would the primary obstruction to finding $k-i+1$ linearly independent sections of E lie? (The Stiefel-Whitney class $w_i(E)$ is the mod-2 reduction of this obstruction.)

(OP-3) Prove that the Euler class, Chern classes, and Stiefel-Whitney classes are natural with respect to pullbacks of vector bundles. E.g., given $f\colon X \to Y$ and $E \to Y$ a vector bundle, $e(f^*E) = f^*e(E)$.

(OP-4) "Show" that the class w_2 which arose when we were considering trivializing the tangent bundles of 3-manifolds in class agrees with the class w_2 defined above.

(OP-5) "Show" that a vector bundle E is orientable if and only if $w_1(E) = 0$. Similarly, show that $c_1(E)$ is the obstruction to finding a complex volume form on E.

(OP-6) Hatcher 4.D.4, 4.D.9, 4.D.10.

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