MATH 636 HOMEWORK 2
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INSTRUCTOR: ROBERT LIPSHITZ

Update April 8: fixed typos in dimensions in problem 1.

Problems to turn in:

(P-1) Given a fiber bundle $p\colon E \to B$ with fiber a sphere S^n, satisfying an orientability condition (see Optional Problem (OP-2)), there is a long exact sequence

$$\cdots \to H^k(B) \longrightarrow H^{n+k+1}(B) \longrightarrow H^{n+k+1}(E) \longrightarrow H^{k+1}(B) \to \cdots,$$

called the Gysin sequence. Here, the map $H^{n+k+1}(B) \to H^{n+k+1}(E)$ is p^* and the map $H^k(B) \to H^{n+k+1}(B)$ is the cup product with a particular cohomology class e.

Suppose that E is the unit sphere bundle in a vector bundle E', $E = SE'$. Suppose further that E' is orientable, so the Thom isomorphism theorem holds. Deduce the existence of the Gysin sequence from the Thom isomorphism theorem. (In the process, you should also describe the cohomology class e.)

(P-2) Show that S^{2n+1} is a fiber bundle over $\mathbb{C}P^n$ with fiber S^1, and in fact this is the sphere bundle associated to an orientable vector bundle. (You do not have to give a detailed proof that the vector bundle you write down is a vector bundle, or orientable.) Then use the Gysin sequence to compute the cohomology ring of $\mathbb{C}P^n$.

(P-3) Given an n-dimensional vector bundle $p\colon E \to B$, let $e(E) \in H^n(B)$ be the cohomology class e from Problem (P-1) associated to the bundle $SE \to B$. Show that if E has a nonvanishing section (or equivalently SE has a section) then $e(E) = 0$. (Hint: this is easy.) Deduce that the vector bundle from Problem (P-2) does not have a nonvanishing section.

(P-4) What does the existence of the Gysin sequence imply about when there are fiber bundles $S^m \to S^n$ with fiber a sphere S^k? (i.e., what relation does it imply among m, n, k?)

(P-5) Prove that, for the category of based topological spaces, the reduced suspension functor is left adjoint to the based loop space functor. (See also Optional Problem (OP-1) for more discussion.)

Review / qualifying exam practice (not to turn in):

(Q-1) Prove the Thom isomorphism theorem.

(Q-2) Let STS^2 be the unit sphere bundle of the tangent bundle of S^2. Use the Mayer-Vietoris sequence to compute the cohomology of STS^2, and deduce that TS^2 has no nonvanishing section.

More problems to think about but not turn in:

(OP-1) In required problem (P-5), you prove that $\mathrm{Map}(\Sigma(X, x_0), (Y, y_0)) \cong \mathrm{Map}((X, x_0), \Omega(Y, y_0))$, naturally. Here, there are two ways you can interpret $\cong$: a bijection of sets, which is easier and what the required problem intended, or as a homeomorphism of topological spaces, where the mapping spaces have the compact-open topology. Prove the stronger result, for the category of based locally compact, Hausdorff spaces. In particular, this implies that $[\Sigma(X, x_0), (Y, y_0)] = [(X, x_0), \Omega(Y, y_0)]$ (for X, Y locally compact, Hausdorff).

(OP-2) Call a sphere bundle $p\colon E \to B$ (with fiber S^n) *orientable* if for every loop $\gamma\colon S^1 \to B$, $\gamma^* E \cong S^1 \times S^n$ (as bundles over S^1). Prove that if $E \to B$ is an orientable sphere bundle then the Thom isomorphism holds for the bundle $DE \to B$ where DE is the mapping cone of the projection $p\colon E \to B$ (so DE is a bundle with fibers D^{n+1}). Deduce that the Gysin exists for any orientable sphere bundle.

(OP-3) Give an example of a non-orientable sphere bundle for which the Gysin sequence does not exist.

Email address: `lipshitz@uoregon.edu`