# MATH 342
# WRITTEN HOMEWORK 1
# DUE APRIL 6, 2020.

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Update 4/1/20: Fixed typo in problems 9 and 10.

*Reminder.* There is *also* online homework. The first online homework is also due Monday, April 6.

Required problems: (hand these in):

(1) Let $S = \{at^2 + b \mid a, b \in \mathbb{R}\}$. Is $S$ a subspace of $\mathbb{P}_2$? Justify your answer.

(2) Let $S = \{at^2 + 2 \mid a \in \mathbb{R}\}$. Is $S$ a subspace of $\mathbb{P}_2$? Justify your answer.

(3) Let $W$ be the set of vectors in $\mathbb{R}^3$ of the form $[2a + b, 3b - a, a + b]^T$. Find vectors $\vec{u}$ and $\vec{v}$ so that $W = \text{Span}\{\vec{u}, \vec{v}\}$. Why does this imply that $W$ is a subspace of $\mathbb{R}^3$?

(4) Let $W$ be the set of vectors in $\mathbb{R}^3$ of the form $[2a+b, 3b-a-1, a+b-2]^T$. Is $W$ a subspace of $\mathbb{R}^3$? If so, find vectors $\vec{u}$ and $\vec{v}$ so that $W = \text{Span}\{\vec{u}, \vec{v}\}$. If not, give an example showing that $W$ is not a subspace.

(5) Complete the following proof that if $c\vec{u} = \vec{0}$ and $c \neq 0$ then $\vec{u} = \vec{0}$:

| | |
|---|---|
| $c\vec{u} = \vec{0}$ | Given. |
| $(1/c)(c\vec{u}) = (1/c)\vec{0}$ | Scalar multiplication is well-defined |
| $((1/c)(c))\vec{u} = (1/c)\vec{0}$ | By axiom ____________(a) |
| $(1)\vec{u} = (1/c)\vec{0}$ | |
| $(1)\vec{u} = \vec{0}$ | By Problem 28 |
| $\vec{u} = \vec{0}$ | By axiom ____________(b) |

(6) Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 7 \end{bmatrix}.$$

Find a (finite) list of vectors which span the null space of $A$.

(7) Let

$$W = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \mid a + c = b + d \right\},$$

with the usual operations of vector addition and scalar multiplication. Use a theorem from class / Section 4.2 to show that $W$ is a vector space.

(8) Let
$$A = \begin{bmatrix} 3 & -6 \\ 2 & -4 \\ 5 & -10. \end{bmatrix}$$
Find a non-zero vector in the column space of $A$ and a non-zero vector in the null space of $A$.

(9) Consider the system of equations
$$\begin{aligned} -1x_1 - 1x_2 + 4x_3 + 5x_4 + 0x_5 - 13x_6 &= 0 \\ -1x_1 - 2x_2 + 8x_3 + 6x_4 - 5x_5 - 7x_6 &= 0 \\ 2x_1 + 8x_2 - 31x_3 - 16x_4 + 27x_5 - 5x_6 &= 0 \\ 3x_1 + 11x_2 - 41x_3 - 22x_4 + 33x_5 + 0x_6 &= 0 \\ 1x_1 + 2x_2 - 7x_3 - 5x_4 + 5x_5 + 4x_6 &= 0 \\ 0x_1 - 5x_2 + 16x_3 + 7x_4 - 5x_5 - 10x_6 &= 0. \end{aligned}$$
It can be shown that $(x_1, x_2, x_3, x_4, x_5, x_6) = (1, -2, -1, -2, -2, -1)$ is a solution. Use material from this week (or Section 4.2 in the book) to explain why $(x_1, x_2, x_3, x_4, x_5, x_6) = (5, -10, -5, -10, -10, -5)$ is another solution. (Do *not* do any nontrivial computations.)

(10) Consider the system of equations
$$\begin{aligned} -1x_1 - 1x_2 + 4x_3 + 5x_4 + 0x_5 &= -13 \\ -1x_1 - 2x_2 + 8x_3 + 6x_4 - 5x_5 &= -7 \\ 2x_1 + 8x_2 - 31x_3 - 16x_4 + 27x_5 &= -5 \\ 3x_1 + 11x_2 - 41x_3 - 22x_4 + 33x_5 &= 0 \\ 1x_1 + 2x_2 - 7x_3 - 5x_4 + 5x_5 &= 4 \\ 0x_1 - 5x_2 + 16x_3 + 7x_4 - 5x_5 &= -10. \end{aligned}$$
It can be shown that this system has a solution. Explain (using recent material) why the system
$$\begin{aligned} -1x_1 - 1x_2 + 4x_3 + 5x_4 + 0x_5 &= 26 \\ -1x_1 - 2x_2 + 8x_3 + 6x_4 - 5x_5 &= 14 \\ 2x_1 + 8x_2 - 31x_3 - 16x_4 + 27x_5 &= 10 \\ 3x_1 + 11x_2 - 41x_3 - 22x_4 + 33x_5 &= 0 \\ 1x_1 + 2x_2 - 7x_3 - 5x_4 + 5x_5 &= -8 \\ 0x_1 - 5x_2 + 16x_3 + 7x_4 - 5x_5 &= 20. \end{aligned}$$
must also have a solution.

(11) Define $T\colon \mathbb{P}_2 \to \mathbb{R}^2$ by $T(p(t)) = \begin{bmatrix} p(1) \\ p(2) \end{bmatrix}$.
  (a) Show that $T$ is a linear transformation.
  (b) Find a polynomial $p(t)$ that spans the kernel of $T$.
  (c) Show that the image (book: range) of $T$ is all of $\mathbb{R}^2$.

(12) Consider the set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \subset \mathbb{R}^3.$$

(a) Does this set span $\mathbb{R}^3$? Justify.
(b) Is this set linearly independent? Justify.
(c) Is this set a basis for $\mathbb{R}^3$? Justify.

(13) Consider the polynomials $p_1(t) = 1+2t$, $p_2(t) = -1+2t$, and $p_3(t) = t$. By inspection, write a linear dependence among $p_1(t)$, $p_2(t)$, and $p_3(t)$. Then find a basis for $\text{Span}(\{p_1(t), p_2(t), p_3(t)\})$.

(14) Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\}$.

(a) Suppose $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$. Find $\vec{x}$.
(b) Let $\vec{y} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$. Find $[\vec{y}]_{\mathcal{B}}$.

(15) Find a basis for the subspace from Problem 3 and find its dimension.

(16) Find the dimensions of the subspaces from Problems 1 and 13.

(17) Let

$$A = \begin{bmatrix} 1 & 2 & -7 \\ 0 & 1 & -4 \\ 2 & 2 & -6 \end{bmatrix}$$

Find bases for $\text{Null}(A)$ and $\text{Col}(A)$ 4.5.14

Suggested practice (don't hand these in):

- Please read and make sure you can do the practice problems in sections 4.1–4.5.
- Please read and use for review problems 4.1.23(a–d), 4.1.24(a–c), 4.2.25, 4.2.26, 4.3.21, 4.3.22, 4.4.15, 4.4.16, 4.5.19, 4.5.20.
- If you had trouble or got help with any of the assigned problems, solve another, similar problem.

Similar problems:

| HW Problems | Similar textbook problems |
|---|---|
| 1–3 | 4.1.5–12 |
| 4 | 4.1.15–18 |
| 5 | 4.1.25–30 |
| 6 | 4.2.3–6 |
| 7 | 4.2.7–14 |
| 8 | 4.2.21, 22 |
| 9, 10 | 4.2.27, 28 |
| 11 | 4.2.31–34 |
| 12 | 4.3.1–8 |
| 13 | 4.3.33, 34 |
| 14 | 4.4.1–8 |
| 15, 16 | 4.5.1–12 |
| 17 | 4.5.13–18 |

*Blog.* Optional: Follow the steps in the post “Getting Started” on the *342 in CoCalc, Spring 2020* blog. The blog is at:
`https://blogs.uoregon.edu/math342sp20lipshitz/`

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