MATH 342
WRITTEN HOMEWORK 2
DUE APRIL 13, 2020.

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Required problems: (hand these in):

(1) Consider the ordered basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$$

(a) Find the change of coordinates matrix from $\mathcal{B}$ to the standard basis $\mathcal{E} = \{\vec{e}_1, \vec{e}_2\}$ for $\mathbb{R}^2$.
(b) Find the change of coordinates matrix from $\mathcal{E}$ to $\mathcal{B}$.
(c) Suppose $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$. Find $\vec{x}$.
(d) Suppose $\vec{y} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Find $[\vec{y}]_{\mathcal{B}}$.

(2) Consider the set $\mathcal{B} = \{1 + t + t^2, t + t^2, t^2\} \subset \mathbb{P}_2$.
(a) Explain why $\mathcal{B}$ forms a basis for $\mathbb{P}_2$, as efficiently as you can.
(b) Find $[2+3t-t^2]_{\mathcal{B}}$, the coordinates of $p(t) = 2+3t-t^2$ with respect to $\mathcal{B}$. Do this two ways: directly and using a change-of-coordinates matrix.

(3) The vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

span $\mathbb{R}^2$ but are not linearly independent. Find two different ways to express $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

(4) Consider the polynomials $p_1(t) = 2 + 3t + 4t^2$, $p_2(t) = 5t + 6t^2$, $p_3(t) = 7t^2$.
(a) Use coordinate vectors to show that $\mathcal{B} = \{p_1(t), p_2(t), p_3(t)\}$ is a basis for $\mathbb{P}_2$.
(b) Suppose that $[q(t)]_{\mathcal{B}} = [1, 0, -1]^T$. Find $q(t)$.

(5) The first four *Chebyshev polynomials of the first kind* are $T_0(t) = 1$, $T_1(t) = t$, $T_2(t) = 2t^2 - 1$, $T_3(t) = 4t^3 - 3t$. (These satisfy $\cos(n\theta) = T_n(\cos(\theta))$; see, e.g., Wikipedia.)
(a) Show that $\mathcal{B} = \{T_0(t), T_1(t), T_2(t), T_3(t)\}$ is a basis for $\mathbb{P}_3$.
(b) Let $p(t) = 1+t+t^2+t^3$. Find the coordinates of $p(t)$ with respect to $\mathcal{B}$.

(6) Here is a matrix and its row-reduced echelon form (thanks, CoCalc):

$$A = \begin{bmatrix} 3 & -11 & 59 & -7 & -25 & -68 & -129 \\ 2 & -7 & 38 & -4 & -16 & -44 & -83 \\ -1 & 8 & -37 & 11 & 18 & 44 & 89 \\ -3 & 17 & -83 & 19 & 32 & 73 & 149 \\ -2 & 12 & -58 & 14 & 29 & 72 & 145 \end{bmatrix} \qquad RREF(A) = \begin{bmatrix} 1 & 0 & 5 & 5 & 0 & 0 & 1 \\ 0 & 1 & -4 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Without doing any calculations, find the rank of A and the nullity of A. Then find bases for the column space of A, the row space of A, and the null space of A.

(7) Suppose a 5×7 matrix A has five pivot columns. Is $\mathrm{Col}(A) = \mathbb{R}^5$? Is $\mathrm{Null}(A) = \mathbb{R}^2$? Explain.

(8) If the null space of a 7×13 matrix is 8-dimensional, what is the dimension of the column space?

(9) Suppose all solutions of a homogeneous system of seven linear equations in eight variables are all multiples of a single non-zero solution. If you change the constants on the right-hand side to some non-zero vector (so the system becomes inhomogeneous), is there necessary a solution? Explain.

(10) Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ and $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ be bases for a vector space V. Suppose $\vec{b}_1 = 2\vec{c}_1 + 3\vec{c}_2$ and $\vec{b}_2 = 3\vec{c}_1 + 4\vec{c}_2$.
  (a) Find the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$, $P_{\mathcal{C}\leftarrow\mathcal{B}}$.
  (b) Find the change of coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$, $P_{\mathcal{B}\leftarrow\mathcal{C}}$.
  (c) Suppose $\vec{x} = 3\vec{c}_1 + 5\vec{c}_2$. Find $[\vec{x}]_{\mathcal{C}}$ and $[\vec{x}]_{\mathcal{B}}$.
  (d) Suppose $\vec{x} = 3\vec{b}_1 + 5\vec{b}_2$. Find $[\vec{x}]_{\mathcal{C}}$ and $[\vec{x}]_{\mathcal{B}}$.

(11) Consider the bases

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\} \qquad \mathcal{C} = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} \right\}$$

for $\mathbb{R}^2$. Find the change of coordinates matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ from $\mathcal{B}$ to $\mathcal{C}$ and the change of coordinates matrix $P_{\mathcal{B}\leftarrow\mathcal{C}}$ from $\mathcal{C}$ to $\mathcal{B}$.

(12) Here is a proof of the book's Theorem 15, which reads:

Theorem. Let $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ and $\mathcal{C} = \{\vec{c}_1, \dots, \vec{c}_n\}$ be bases for a vector space V. Then there is a unique $n \times n$ matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ so that for any $\vec{x} \in V$,

$$[\vec{x}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[\vec{x}]_{\mathcal{B}}. \tag{1}$$

Further,

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \left[[\vec{b}_1]_{\mathcal{C}} \cdots [\vec{b}_n]_{\mathcal{C}} \right]. \tag{2}$$

Fill in the blanks in the following proof:

Fix a vector $\vec{v} \in V$. Since (a)____________, there are real numbers $x_1, \dots, x_n$ so that

$$\vec{v} = x_1\vec{b}_1 + \cdots + x_n\vec{b}_n.$$

Applying the coordinate mapping associated to $\mathcal{C}$ to both sides, this gives

$$[\vec{v}]_{\mathcal{C}} = x_1[\vec{b}_1]_{\mathcal{C}} + \cdots + x_n[\vec{b}_n]_{\mathcal{C}}.$$

This uses the fact that the coordinate mapping is (b)____________. By the definition of (c)____________, this equation can be rewritten as

$$[\vec{v}]_{\mathcal{C}} = \begin{bmatrix}[\vec{b}_1]_{\mathcal{C}} \cdots [\vec{b}_n]_{\mathcal{C}}\end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Notice that the vector $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is equal to (d)____________ Hence, the matrix in Equation (2) satisfies Equation (1).

Now, suppose Q is any other matrix so that for any $\vec{x} \in V$,

$$[\vec{x}]_{\mathcal{C}} = Q[\vec{x}]_{\mathcal{B}}. \tag{3}$$

Take $\vec{x} = \vec{b}_i$. Then $[\vec{x}]_{\mathcal{B}}$ is equal to (e)____________, so the right side of Equation (3) is equal to (f)____________. Since this is true for every i, this proves that Q is equal to the matrix from Equation (2).

(13) (a) Show that the two elements (signals) $\{(-2)^k\}_{k\in\mathbb{Z}}, \{3^k\}_{k\in\mathbb{Z}} \in \mathbb{S}$ are solutions of the difference equation

$$y_{k+2} - y_{k+1} - 6y_k = 0.$$

(b) Show that these two signals $\{(-2)^k\}_{k\in\mathbb{Z}}, \{3^k\}_{k\in\mathbb{Z}}$ are linearly independent in $\mathbb{S}$.

(c) Show that these two signals $\{(-2)^k\}_{k\in\mathbb{Z}}, \{3^k\}_{k\in\mathbb{Z}}$ span the space of solutions to

$$y_{k+2} - y_{k+1} - 6y_k = 0.$$

hence form a basis for it.

(14) Show that the signal $y_k = k^2$ is a solution to the inhomogeneous difference equation

$$y_{k+2} + 2y_{k+1} - 3y_k = 8k + 6.$$

Then find the general solution to this inhomogeneous difference equation.

(15) Find a steady state vector for the matrix

$$\begin{bmatrix} 1/4 & 5/6 \\ 3/4 & 1/6 \end{bmatrix}.$$

(16) (cf. problem 4.9.3 in the textbook) Here is a questionable model for infection. On any given day, each person is either healthy or sick. Of the healthy people, 95% will be healthy tomorrow, and the remainder will be sick. Of the sick people, 60% will still be sick tomorrow, and the remainder will be healthy.

(a) Write the stochastic matrix representing this situation.

(b) If 10% of people are sick today, what fraction will be sick tomorrow? The day after tomorrow?
(c) If a student is healthy today, what is the chance s/he will be healthy in two days?
(d) Find a steady-state vector for this model. What do its entries represent?
(e) What are some problems with this model, in general?
(f) Can you describe a situation or disease where this model (perhaps with different numbers) seems relatively reasonable?

Suggested practice (don't hand these in):

- Please read and make sure you can do the practice problems in sections 4.6–4.9.
- Please read and use for review problems 4.6.17(a,c,d,e), 4.6.18, 4.7.11, 4.7.12.
- If you had trouble or got help with any of the assigned problems, solve another, similar problem.

Similar problems:

HW Problems	Similar textbook problems
1	4.4.9–12
2	4.4.13–14, 4.7.12–14
3	4.4.17
4	4.4.31, 4.4.32
5	4.4.21–24
6	4.6.1–4
7, 8	4.6.5–16
9	4.6.19–25
10	4.7.1–6
11	4.7.7–10
12	4.7.15–16
13	4.8.1–12
14	4.8.25–28
15	4.9.5–8
16	4.9.1–4, 12

Blog. Optional: Follow the steps in the post "Matrix Operations in CoCalc" on the *342 in CoCalc, Spring 2020* blog. The blog is at: `https://blogs.uoregon.edu/math342sp20lipshitz/`

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