MATH 342  
WRITTEN HOMEWORK 4  
INSTRUCTOR: ROBERT LIPSHITZ

Required problems (hand these in):

(1) Let $V$ be a vector space and $B = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ an ordered basis for $V$. Suppose that $T: V \to \mathbb{R}^2$ is a linear transformation so that

$$T(x_1 \vec{b}_1 + x_2 \vec{b}_2 + x_3 \vec{b}_3) = \begin{bmatrix} x_1 - x_2 + 2x_3 \\ 3x_1 - 5x_2 \end{bmatrix}.$$  

Find the matrix for $T$ with respect to $B$ and the standard basis for $\mathbb{R}^2$.

(2) Let $T: P_2 \to P_4$ be the transformation $T(p(t)) = (t + 3)p(t) + p'(t)$.

(a) Compute $T(2 + 3t + 4t^2)$.

(b) Show that $T$ is a linear transformation.

(c) Find the matrix for $T$ with respect to the ordered bases $\{1, t, t^2\}$ for $P_2$ and $\{1, t, t^2, t^3\}$ for $P_3$.

(3) Define $T: P_1 \to \mathbb{R}^2$ by

$$T(p(t)) = \begin{bmatrix} p(0) \\ p(1) \\ p(2) \end{bmatrix}.$$  

(a) Compute $T(2 + 3t)$.

(b) Show that $T$ is a linear transformation.

(c) Find the matrix for $T$ with respect to the ordered basis $\{1, t\}$ for $P_1$ and the standard basis for $\mathbb{R}^3$.

(4) Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T(\vec{x}) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \vec{x}.$$  

Let

$$B = \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}.$$  

Find the matrix $[T]_B$ for $T$ with respect to $B$.

(5) Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T(\vec{x}) = \begin{bmatrix} -7 & 18 \\ -3 & 8 \end{bmatrix} \vec{x}.$$  

Find a basis $B$ so that $[T]_B$ is diagonal (and find $[T]_B$).
(6) Let \( V = \{ a_0 + a_1 e^x + a_2 e^{2x} + a_3 e^{3x} \mid a_1, a_2, a_3 \in \mathbb{R} \} \), which is a subspace of the vector space of functions \( \mathbb{R} \to \mathbb{R} \). Let \( T: V \to V \) be the linear transformation \( T(f(x)) = f''(x) \). What are the eigenvalues of \( T \)? Find an eigenvector corresponding to each eigenvalue.

(7) Consider the linear transformation \( T: \mathbb{P}_2 \to \mathbb{P}_2 \) given by \( T(p(x)) = p(0) + p(1) + p'(x) + 3x^2p''(x) \). Let \( \mathcal{B} \) be the basis \( \{1, x, x^2\} \) for \( \mathbb{P}_2 \).
(a) Find the matrix \( A \) for \( T \) with respect to the basis \( \mathcal{B} \).
(b) Find the eigenvalues of \( A \), and a basis for \( \mathbb{R}^3 \) consisting of eigenvectors of \( A \).
(c) Find a basis for \( \mathbb{P}_2 \) consisting of eigenvectors for \( T \).

(8) Let
\[
A = \begin{bmatrix}
2 & -1 \\
1 & 2
\end{bmatrix}.
\]
Find the eigenvalues of \( A \) (which may be complex numbers) and a basis for each eigenspace in \( \mathbb{C}^2 \).

(9) Let
\[
A = \begin{bmatrix}
3 & -2 \\
1 & 1
\end{bmatrix}.
\]
Find the eigenvalues of \( A \) (which may be complex numbers) and a basis for each eigenspace in \( \mathbb{C}^2 \).

(10) The matrix
\[
\begin{bmatrix}
2 & -1 \\
1 & 2
\end{bmatrix}
\]
corresponds to the composition of a rotation and a scaling. Give the angle \( \phi \) of the rotation and the scale factor \( r \).

(11) Let \( A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \). Compute \( A^{20} \) (by hand).

(12) Plot the following complex numbers in the complex plane, on one graph: \( 1 + i \), \( 2 - 3i \), \( 2e^{\pi i} \), \( 2e^{\frac{2\pi}{3}i} \), \( 2 \cos(\pi/3) + 2i \sin(\pi/3) \).

(13) Plot the following complex numbers in the plane, on one graph: \( (1 + i\sqrt{3})^2 \), \( (1 + i\sqrt{3})^3 \), \( (1 + i\sqrt{3})^4 \), \( e^{\pi i} \), \( e^{\frac{2\pi}{3}i} \), \( e^{\frac{4\pi}{3}i} \), \( e^{\frac{5\pi}{3}i} \).

Suggested practice (don’t hand these in):
- Please read and make sure you can do the practice problems in Sections 5.4 and 5.5.
- If you have taken a course on writing proofs, try problems 5.4.25, 5.4.26, 5.4.19, 5.4.20, 5.4.21, 5.4.23, 5.5.23, and 5.5.24.
- If you had trouble or got help with any of the assigned problems, solve another, similar problem (or two).
Similar problems:

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Blog. Optional:
- Work through the blog post “Complex numbers”.
- Use CoCalc to check your answers to problems 4, 8, 9, 12, and 13.

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