MATH 342
WRITTEN HOMEWORK 5
DUE MAY 4, 2020.

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Note that WebWorks and the written homework are due at *different* times this week.

Chapter 10 is an online supplement to the textbook, which you can download from the publisher.

Required problems (hand these in):

(1) Let A be a 3×3 matrix with eigenvalues 2, 2/3, and 1/4, and corresponding eigenvectors

$$\begin{bmatrix}1\\2\\3\end{bmatrix}, \quad \begin{bmatrix}1\\0\\1\end{bmatrix}, \text{ and } \begin{bmatrix}2\\1\\0\end{bmatrix}$$

Let $\vec{x}_0 = \begin{bmatrix}0\\1\\4\end{bmatrix}$. Find a closed-form solution to the equation $\vec{x}_{k+1} = A\vec{x}_k$ and describe what happens as $k \to \infty$.

(2) In a city inhabited by people and zombies, zombies eat people. The predator-prey matrix for these populations is

$$\begin{bmatrix}.1 & 1\\ -p & 1.3\end{bmatrix}$$

(where we represent the population as [people, zombies]^T).

(a) Show that if the predation parameter p is .2 then for suitable initial values, both populations grow. Estimate the long-term growth rate and eventual ratio of people to zombies.

(b) For other initial values, this model gives garbage answers. For example, what happens after one time unit if the city starts out with 1000 people and 0 zombies? Why is this nonsense?

(c) Show that if the predation parameter p is .4, then eventually both humans and zombies die off (no matter what the initial values are).

(3) For each of the following matrices, classify the origin as either an attractor, repeller, or saddle point of the dynamical system $\vec{x}_{k+1} = A\vec{x}_k$, and find the directions of greatest attraction and/or repulsion:

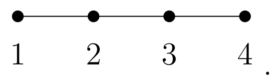
(a)

$$A = \begin{bmatrix}7/4 & -5/4\\ 1/4 & 1/4\end{bmatrix}$$

(b)

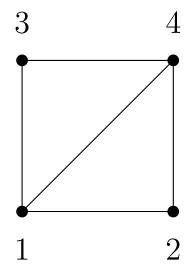
$$A = \begin{bmatrix} 3/16 & 5/16 \\ -1/16 & 9/16 \end{bmatrix}$$

(4) Consider an unbiased random walk on the set $\{1, 2, 3, 4\}$, i.e., on the graph

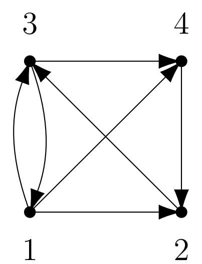


What is the probability of moving from 2 to 3 in exactly 3 steps if the walk has:
(a) reflecting boundaries?
(b) absorbing boundaries?

(5) Find the transition matrix for the simple random walk on the following graph:



(6) Find the transition matrix for the simple random walk on the following directed graph:



(7) Consider the stochastic matrix, for a Markov process with two states:

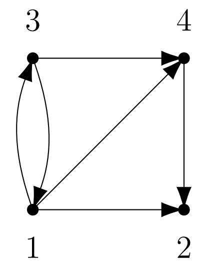
$$P = \begin{bmatrix} 1/2 & 3/4 \\ 1/2 & 1/4 \end{bmatrix}.$$

Find the probability that, in the long run, the process is in state 1, in two ways. First, raise P to a high power. (You may use software if you like.) Second, compute the steady-state vector (by computing the null space of $P - I$).

(8) Consider the unbiased random walk from Problem 4a. Explain why this transition matrix is not regular (using the definition of regular). Then explain why Theorem 1(a) in Section 10.2 ("There is a stochastic matrix Π such that $\lim_{n\to\infty} P^n = \Pi$") is false for this matrix.

(9) Find the Google matrix for the following directed graph, and compute the PageRank score of each vertex and the PageRank ordering of the

vertices. Use the parameter $p = .85$. Feel free to use computer software to do the computations.



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(10) Find a unit vector in the direction of $[1, 2, 3]^T$.
(11) Are the vectors $[1, 2]^T$ and $[-4, 3]^T$ orthogonal? (Justify briefly.)
(12) Are the vectors $[1, 2]^T$ and $[-4, 2]^T$ orthogonal? (Justify briefly.)

Suggested practice (don't hand these in):

- Please read and make sure you can do the practice problems in Sections 5.6, 10.1, 10.2, and 6.1.
- Use Exercises 10.1.21, 10.1.22, 10.2.21, 10.2.22, 6.1.19, 6.1.20, and 6.1.22 for review.
- If you have taken a course on writing proofs, try problems 6.1.27, 6.1.28, and 6.1.31.
- If you had trouble or got help with any of the assigned problems, solve another, similar problem (or two).

Similar problems:

HW Problems	Similar textbook problems
1	5.6.1–2
2	5.6.5–6
3	5.6.9–14
4	10.1.11–12
5	10.1.13–14
6	10.1.15–16
7	10.2.1–6
8	10.2.11–12
9	10.2.25–26
10	6.1.9–12
11–12	6.1.15–16

Blog. Optional: No new entry this week, but:

- Use CoCalc to check your answers to the problems above.
- Use CoCalc to check the computation of the steady state vector $\vec{q}$ on page 21 of section 10.2. What power of G do you have to take in order to get the vector $\vec{q}$ to 6 digits of accuracy?
- Bonus bonus points: figure out how to get Sage to produce a random-ish 100×100 stochastic matrix. Then find the steady-state vector of the associated "Google matrix" to 5 digits of accuracy.

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