Required problems (hand these in):

(1) Determine whether each of the following sets of vectors is orthonormal, orthogonal but not orthonormal, or not orthogonal.
   (a) \[
   \begin{bmatrix}
   1 \\
   1 \\
   1 \\
   \end{bmatrix}, \begin{bmatrix}
   1 \\
   0 \\
   -1 \\
   \end{bmatrix}, \begin{bmatrix}
   0 \\
   0 \\
   0 \\
   \end{bmatrix}
   \]
   (b) \[
   \begin{bmatrix}
   2/3 \\
   2/3 \\
   1/3 \\
   \end{bmatrix}, \begin{bmatrix}
   -2/3 \\
   1/3 \\
   2/3 \\
   \end{bmatrix}, \begin{bmatrix}
   1/3 \\
   -2/3 \\
   2/3 \\
   \end{bmatrix}
   \]
   (c) \[
   \begin{bmatrix}
   1 \\
   1 \\
   1 \\
   \end{bmatrix}, \begin{bmatrix}
   1 \\
   0 \\
   -1 \\
   \end{bmatrix}, \begin{bmatrix}
   0 \\
   1 \\
   0 \\
   \end{bmatrix}
   \]

(2) Compute the orthogonal projection of \( \vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \) onto the line \( L \) through \( \begin{bmatrix} -1 \\ 4 \end{bmatrix} \) and the origin. Then write \( \vec{v} \) as the sum of a vector in \( L \) and a vector orthogonal to \( L \).

(3) Consider the pair of vectors \( \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \).
   (a) Normalize these vectors to obtain an orthonormal set.
   (b) Find the orthogonal projection of \( \vec{v} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \) onto the subspace \( \text{Span}(\{\vec{u}_1, \vec{u}_2\}) \).

(4) Let \( \vec{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \).
(a) Find the closest point to \( \vec{y} \) in \( \text{Span}(\{\vec{v}_1, \vec{v}_2\}) \).

(b) Write \( \vec{y} \) as the sum of a vector in \( \text{Span}(\{\vec{v}_1, \vec{v}_2\}) \) and a vector orthogonal to \( \text{Span}(\{\vec{v}_1, \vec{v}_2\}) \).

(c) Find the best approximation to \( \vec{y} \) by vectors of the form \( c_1 \vec{v}_1 + c_2 \vec{v}_2 \).

What are \( c_1 \) and \( c_2 \)?

(5) Consider the subspace

\[
W = \text{Span} \left\{ \begin{bmatrix} 4 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 7 \end{bmatrix} \right\} \subset \mathbb{R}^3.
\]

Apply the Gram-Schmidt process to find an orthonormal basis for \( W \).

(6) Find an orthonormal basis for the column space of the following matrix:

\[
\begin{pmatrix}
4 & 3 & 5 \\
2 & 4 & 5 \\
2 & 0 & -4 \\
1 & 5 & 3
\end{pmatrix}
\]

(7) Let \( \{\vec{v}_1, \ldots, \vec{v}_k\} \) be an orthonormal basis for a subspace \( W \) of \( \mathbb{R}^n \). Define \( T: \mathbb{R}^n \to \mathbb{R}^n \) to be \( T(\vec{x}) = \text{proj}_W(\vec{x}) \). Show that \( T \) is a linear transformation. (Hint: start by writing down the formula for \( T \).)

Suggested practice (don’t hand these in):

- Please read and make sure you can do the practice problems in Sections 6.2, 6.3, and 6.4.
- Use Exercises 6.2.23, 6.2.24, 6.3.21, 6.3.22, 6.4.17(a,b), 6.4.18(a,b) for review.
- If you have some experience writing proofs, try Exercises 6.2.28, 6.2.29, 6.2.33.
- If you had trouble or got help with any of the assigned problems, solve another, similar problem (or two).
Similar problems:

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Blog. Optional:

- Work through the blog post “Dot Products and Orthogonality”.
- Use CoCalc to check your answers to the problems above (except for Problem 7).
- Write a function in CoCalc (Sage) which computes the orthogonal projection of \( \vec{v} \) to \( \text{Span}(\vec{w}) \), where \( \vec{v} \) and \( \vec{w} \) are vectors in the same \( \mathbb{R}^n \). (Your function should take two inputs, \( \vec{v} \) and \( \vec{w} \).) Test your function in some examples. Only slightly harder: write a function which computes the orthogonal projection of \( \vec{v} \) onto \( \text{Span}(\vec{w}_1, \ldots, \vec{w}_k) \). (In the second case, start by applying Sage’s Gram-Schmidt to \( \vec{w}_1, \ldots, \vec{w}_k \).)

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