MATH 342
WRITTEN HOMEWORK 7
DUE MAY 18, 2020.

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Note. I didn't assign many problems from section 6.7, because there are several on the WebWorks. You might like to solve some of 3–8 for additional practice similar to the WebWorks problems.

Required problems (hand these in):

(1) Find a least-squares solution of $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 2 & 0 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},$$

in two ways:

(a) Use what the book calls the "normal equations" for the least squares solution, involving A^T.

(b) Find the orthogonal projection $\mathbf{c}$ of $\mathbf{b}$ to the column space of A and then solve $A\mathbf{x} = \mathbf{c}$.

(Note: the answer is not especially pretty.)

(2) Compute the least-squares error associated to the solution you found in Problem 1 (how close $A\mathbf{x}$ is to $\mathbf{b}$).

(3) Consider the space $\mathbb{P}_2$ with the inner product

$$\langle p, q \rangle = \frac{1}{2} \int_{-1}^{1} p(t)q(t)dt.$$

Apply the Gram-Schmidt process to $\{1, t, t^2\}$ to find an orthonormal basis for $\mathbb{P}_2$ with respect to these polynomials. (The answers are constant multiples of the first three Legendre polynomials. The Chebyshev polynomials from Written Homework 2 are also an orthogonal set, with respect to a different inner product (but the computation is more tedious).)

(4) Consider the vector space $\mathcal{C}[0, 2\pi]$ of continuous function $[0, 2\pi] \to \mathbb{R}$ with the inner product

$$\langle f, g \rangle = \int_0^{2\pi} f(x)g(x)dx.$$

Show that $\sin(mt)$ and $\cos(nt)$ are orthogonal for all positive integers m, n, and $\sin(mt)$ and $\sin(nt)$ are orthogonal if $m \neq n$.

(5) Find the first-order Fourier approximation to $f(t) = t+1$, i.e., the best approximation to $f(t)$ by functions in

$$\text{Span}\{1, \cos(t), \sin(t)\}$$

with respect to the inner product from Problem 4.

(6) Let A be a 2×2 matrix with eigenvalues 2 and 3 and corresponding eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Find the solution to $\mathbf{x}'(t) = A\mathbf{x}(t)$ with initial value $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$.

(7) Consider the differential equation $\mathbf{x}'(t) = A\mathbf{x}(t)$ where

$$A = \begin{bmatrix} 1 & 7 \\ 3 & 5 \end{bmatrix}.$$

(a) Diagonalize A, as $A = PDP^{-1}$.
(b) Write $\mathbf{x}(t) = P\mathbf{y}(t)$ and show (by plugging in) that this decouples the differential equation.
(c) Find the general solution to the differential equation $\mathbf{y}'(t) = D\mathbf{y}(t)$.
(d) Find the general solution to the differential equation $\mathbf{x}'(t) = A\mathbf{x}(t)$.
(e) Is the origin an attractor, repeller, or saddle point for this dynamical system? Find the directions of greatest attraction / repulsion, and make a rough sketch of what typical trajectories look like.

(8) Consider the differential equation $\mathbf{x}'(t) = A\mathbf{x}(t)$ where

$$A = \begin{bmatrix} 3 & 4 \\ -5 & -5 \end{bmatrix}.$$

(a) Find the general solution using complex eigenfunctions (i.e., $e^{(a+bi)t}$).
(b) Find the general real solution (in terms of functions $e^{at}\sin(bt)$ and $e^{at}\cos(bt)$).
(c) Describe the rough shape of typical trajectories.

Suggested practice (don't hand these in):

- Please read and make sure you can do the practice problems in Sections 6.2, 6.3, and 6.4.
- Use Exercises 6.2.23, 6.2.24, 6.3.21, 6.3.22, 6.4.17(a,b), 6.4.18(a,b) for review.
- If you have some experience writing proofs, try Exercises 6.2.28, 6.2.29, 6.2.33.
- If you had trouble or got help with any of the assigned problems, solve another, similar problem (or two).

Similar problems:

HW Problems	Similar textbook problems
1	6.5.1–4, 9–12
2	6.5.7–8
3	6.7.25–26. Maybe also 6.7.21–24.
4	6.8.5–7
5	6.8.8–9
6	5.7.1–2
7	5.7.3–8
8	5.7.9–14

Blog. Optional:

- Work through the short blog entries "Least-squares", "Fourier series", and "Differential Equations".
- Try our `numerical_fourier` function on $f(x) = x$. Plot a few of the results. You should see that it converges very slowly. This is because the function $f(x) = x$ is not really periodic, i.e., the values at $-\pi$ and π do not match.
- Use CoCalc to plot solutions to the dynamical systems in Problems 7 and 8.

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