MATH 636 HOMEWORK 1
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Updated: typo fixed in (4b).

(1) Hatcher 3.3.24 (p. 259). Deduce also that if \( M \) is simply connected then \( H_i(M) \cong H_i(S^3) \) for each \( i \).

(2) Hatcher 3.3.31 (p. 260).

(3) Hatcher 3.3.32 (p. 260).

(4) Let \( X \) be a closed, orientable \( 4n \)-dimensional manifold.

(a) Recall that the signature of a bilinear form is the number of positive eigenvalues minus the number of negative eigenvalues. Prove the following algebraic lemma.

**Lemma.** Let \( V \) be a \( 2k \)-dimensional \( \mathbb{R} \)-vector space and \( g: V \times V \to \mathbb{R} \) a nondegenerate, symmetric, bilinear form. If there is a \( k \)-dimensional subspace \( W \subset V \) so that \( g(v, w) = 0 \) for all \( v, w \in W \) then the signature of \( g \) is zero.

(b) Fix an orientation \([X]\) for \( X \). Define the signature \( \sigma(X) \) of \( X \) to be the signature of the bilinear form \( H^{2n}(X; \mathbb{R}) \times H^{2n}(X; \mathbb{R}) \to \mathbb{R} \)

\[ (a, b) \mapsto \langle a \cup b, [X] \rangle. \]

Show that if \( X \) is the boundary of a compact, orientable \((4n+1)\)-dimensional manifold-with-boundary then \( \sigma(X) = 0 \).

(c) Deduce that \( \mathbb{C}P^{2n} \) is not the boundary of an orientable \((4n+1)\)-dimensional manifold.

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Suggested review / qualifying exam practice (not to turn in):

(1) Use Lefschetz duality to show that the cup product

\[ H^k(D^k \times D^\ell, S^{k-1} \times D^\ell) \otimes H^\ell(D^k \times D^k, D^k \times S^{\ell-1}) \to H^{k+\ell}(D^k \times D^\ell, \partial(D^k \times D^\ell)) \]

is an isomorphism. (This is a lemma we proved earlier, by more direct techniques.)

(2) A 2-component link is a pair of disjoint, smooth embeddings \( K_1, K_2: S^1 \hookrightarrow S^3 \). Given a 2-component link, the linking number of \( L \) is defined as follows:

(a) By Alexander duality, \( H_1(S^3 \setminus K_1(S^1)) \cong H^1(K_1(S^1)) \cong H^1(S^1) \cong \mathbb{Z} \), where the last isomorphism sends the positive generator of \( H^1(K_1) \) to 1. On the other hand, \( (K_2)_*([S^1]) \) is an element of \( H_1(S^3 \setminus K_1(S^1)) \). The image of this element in \( \mathbb{Z} \) is the linking number of the link.

(b) Alternatively, let nbd\((K_1(S^1))\) be a small solid torus neighborhood of \( K_1(S^1) \). Then \( H_2(S^3 \setminus \text{nbd}(K_1(S^1)), \partial\text{nbd}(K_1(S^1))) \to H_1(\partial\text{nbd}(K_1(S^1))) \to H_1(K_1(S^1)) \) is an isomorphism (this needs proof), so the generator \( (K_1)_*([S^1]) \) gives a preferred generator of \( H_2(S^3 \setminus \text{nbd}(K_1(S^1)), \partial\text{nbd}(K_1(S^1))) \). By Lefschetz duality,
Let $p$.

Given $n$.

More problems to think about but not turn in:

(1) Let $Y$ be a compact, orientable 3-manifold-with-boundary, and $F = \partial Y$, a closed surface of genus $g$.

(a) Fix an orientation (fundamental class) $[F]$ for $F$. Define a pairing $\omega: H^1(F) \times H^1(F) \to \mathbb{Z}$ by

$$\omega(a, b) = \langle a \cup b, [F] \rangle = (a \cup b) \cap [F].$$

Prove that $\omega$ is skew-symmetric and non-degenerate.

(b) Let $K \subset H^1(F)$ be the image of $j^*: H^1(Y) \to H^1(F)$ (where $j$: $F \to Y$ is inclusion). Show that for any $a, b \in K$, $\omega(a, b) = 0$.

(c) Deduce from Hatcher, problem 3.3.28, that the rank of $K$ is at most $g$. In particular, deduce that if $g > 0$ then $Y$ is not simply connected.

(d) Show that the rank of $K$ is exactly $g$. (In other words, $K$ is a Lagrangian subspace of $(H^1(F), \omega)$.)

(2) Hatcher 3.3.35.

(3) Read through Hatcher’s first computation of the ring structure on $H^*(\mathbb{C}P^n)$ with Poincaré duality in mind, and see how he came up with it.

(4) Let $Y$ be a closed, smooth, oriented 3-manifold. Show that every homology class $\alpha \in H_2(Y; \mathbb{Z})$ is represented by some embedded surface $\Sigma \subset Y$, as follows. Let $b = \text{PD}(\alpha) \in H^1(Y; \mathbb{Z})$. From last quarter, there is a map $f: Y \to S^1$ so that $f^*(d\theta) = b$, where $d\theta \in H^1(S^1; \mathbb{Z})$ is the preferred generator. (Here, perhaps we are using the fact that $Y$ is homeomorphic to a CW complex.) Show that $f$ is homotopic to a smooth map $g: Y \to S^1$. Let $p \in S^1$ be a regular value of $g$. Show that $\Sigma = g^{-1}(p)$ represents $\alpha$.

(5) Let $p(z_0, z_1, z_2), q(z_0, z_1, z_2) \in \mathbb{C}[z_0, z_1, z_2]$ be degree $d$ homogeneous polynomials so that, viewing $p$ and $q$ as maps $\mathbb{R}^6 \cong \mathbb{C}^3 \to \mathbb{C} \cong \mathbb{R}^2$, 0 is a regular value of $p$ and $q$.

(a) Show that $\{(z_0 : z_1 : z_2) \in \mathbb{CP}^2 | p(z_0, z_1, z_2) = 0\}$ is a smooth surface $\Sigma_p$.

(b) Show that there is a smooth path $p_t$ of homogeneous polynomials so that $p_0 = p$ and $p_1 = q$, and 0 is a regular value of $p_t$ for all $t$. (Hint: show that 0 not being a regular value is a polynomial equation on the coefficients of $p_t$, and hence has real codimension 2.)

(c) Show that $\Sigma_p$ and $\Sigma_q$ represent the same element of $H_2(\mathbb{CP}^2)$, as follows. Let $p_t$ be as in the previous part. Then

$$\{(t, z_0 : z_1 : z_2) \in [0, 1] \times \mathbb{CP}^2 | p_t(z_0, z_1, z_2) = 0\}$$

is an orientable 3-manifold inside $[0, 1] \times \mathbb{CP}^2$, with boundary $-\Sigma_p \bigsqcup \Sigma_q$ (where the $-$ denotes orientation reversal). Hence, from an exercise last week, $[\Sigma_p] = [\Sigma_q] \in H_2([0, 1] \times \mathbb{CP}^2) = H_2(\mathbb{CP}^2)$.

(6) Given $n \in \mathbb{N}$, let $Y_n = \{z \in \mathbb{C} | z^n \in \mathbb{R}_{\geq 0}\}$, so $Y_n$ looks like a graph with one central vertex and $n$ spokes. A branched surface is a topological space $X$ so that each point $x \in X$ has a neighborhood $U \ni x$ homeomorphic to either $\mathbb{R}^2$ or $(0, 1) \times Y_n$ for some $n$. (That is, near each point $Y$ either looks like $\mathbb{R}^2$ or like $n$ half-planes coming together.) A $p$-branched surface is a branched surface where the only $Y_n$’s which occur have $n$
divisible by $p$. (So, for instance, a 2-branched surface can look locally like a plane, like 4 half-planes coming together, like 6 half-planes coming together, and so on.)

(a) Suppose $X$ is a compact 2-branched surface. Show that $X$ has a fundamental class $[X] \in H_2(X; \mathbb{F}_2)$.

(b)

(c) (Harder) Read what the Bockstein homomorphism is and relate it to representing mod-2 homology classes by 2-branched surfaces.

(d) (Harder) Generalize the previous two parts to $p$-branched surfaces for $p$ an odd prime. Here, you will need a notion of orientability for $p$-branched surfaces.

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