MATH 636 HOMEWORK 10
DUE JUNE 4, 2021.

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(1) Use the path-loop fibration to compute the homology of ΩS^n for $n \geq 2$. Comment also on the case $n = 1$.
(2) Explain how the group $SU(3)$ acts transitively on S^5 so that the stabilizer of a point is $SU(2)$. It follows that there is a fibration $SU(2) \to SU(3) \to S^5$; this is a standard result which you do not have to prove this (see, e.g., Exercise 21.6 in Lee's *Introduction to Smooth Manifolds*, and/or chapter 24 there). Use this fibration and the fact that $SU(2) \cong S^3$ (which you may again assume) to compute the homology of $SU(3)$. Then use the fibration $SU(3) \to SU(4) \to S^7$ to compute the homology of $SU(4)$.
(3) Suppose $F \to E \to S^n$, $n \geq 2$, is a fibration. Show that for any field k there is a long exact sequence

$$\cdots \to H_k(F) \to H_k(E) \to H_{k-n}(F) \to H_{k-1}(F) \to \cdots .$$

(This is called the *Wang sequence*, and is due to Hsien-Chung Wang from 1949. The assumption that we are working over a field is unnecessary, but saves you a little algebra.)
(4) Show that if every term in the E_∞-page of the Serre spectral sequence for a fibration $F \to E \to B$ is a finite abelian group then $H_i(E)$ is a finite abelian group for each i. (Hint: this is easy.)
(5) Let $\mathscr{C}$ be the class of torsion-free abelian groups. Find a space X so that $H_*(X) \in \mathscr{C}$ but $\pi_*(X) \notin \mathscr{C}$. Find another space Y so that $\pi_*(Y) \in \mathscr{C}$ but $H_*(Y) \notin \mathscr{C}$. So, Serre's mod-$\mathscr{C}$ Hurewicz theorem definitely does not apply to $\mathscr{C}$. Where does the proof break down?

Suggested review / qualifying exam practice (not to turn in):

(1) Extend our computation of $H^n(K(\mathbb{Z},3))$ for $n \leq 7$ to compute $H^8(K(\mathbb{Z},3))$ and $H^9(K(\mathbb{Z},3))$. (In fact, you can keep going for a while longer than this before you get stuck.)
(2) Use the method of canceling arrows to compute the homology of some familiar simplicial complexes—for example, some triangulations of T^2 or the Klein bottle.
(3) Extend Problem 4 to the other Serre classes we introduced.
(4) Use the Serre spectral sequence and the path-loop fibration to deduce the n-dimensional Hurewicz theorem from the 1-dimensional case. (This is about the original Hurewicz theorem, not the mod-$\mathscr{C}$ version.)

More problems to think about but not turn in:

(1) Along the lines of Problem 2, a standard application of the Serre spectral sequence to inductively prove that $H^*(SU(n)) \cong \Lambda(x_2, \dots, x_n)$ where $x_i \in H^{2i-1}(SU(n))$. This is easy if you assume that the ring structure on the E_∞-page of the Serre spectral sequence for $SU(n) \to SU(n+1) \to S^{2n+1}$ is isomorphic to the ring structure on $H^*(SU(n+1))$. Prove the result assuming that.

Slightly harder is to prove that the fact you used about the ring structure, using the fact that the ring structure on the E_∞-page is the associated graded to a filtration on $H^*(SU(n+1))$. Prove that, too, or see for example McCleary's book.

(2) Strengthen our proofs of the Gysin and Wang sequences to work with $\mathbb{Z}$-coefficients. (This is an algebra problem.)

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