

**MATH 636 HOMEWORK 2**  
**DUE APRIL 9, 2021.**

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- (1) Given a fiber bundle  $p: E \rightarrow B$  with fiber a sphere  $S^n$ , satisfying an orientability condition (see Optional Problem 1), there is a long exact sequence

$$\cdots \rightarrow H^k(B) \rightarrow H^{n+k+1}(B) \rightarrow H^{n+k+1}(E) \rightarrow H^{k+1}(B) \rightarrow \cdots,$$

called the Gysin sequence. Here, the map  $H^{n+k+1}(B) \rightarrow H^{n+k+1}(E)$  is  $p^*$  and the map  $H^k(B) \rightarrow H^{n+k+1}(B)$  is the cup product with a particular cohomology class  $e$ .

Suppose that  $E$  is the unit sphere bundle in a vector bundle  $E'$ ,  $E = SE'$ . Suppose further that  $E'$  is orientable, so the Thom isomorphism theorem holds. Deduce the existence of the Gysin sequence from the Thom isomorphism theorem. (In the process, you should also describe the cohomology class  $e$ .)

- (2) Show that  $S^{2n+1}$  is a fiber bundle over  $\mathbb{C}P^n$  with fiber  $S^1$ , and in fact this is the sphere bundle associated to an orientable vector bundle. (You do not have to give a detailed proof that the vector bundle you write down is a vector bundle, or orientable.) Then use the Gysin sequence to compute the cohomology ring of  $\mathbb{C}P^n$ .
- (3) Given an  $n$ -dimensional vector bundle  $p: E \rightarrow B$ , let  $e(E) \in H^n(B)$  be the cohomology class  $e$  from Problem 1 associated to the bundle  $SE \rightarrow B$ . Show that if  $E$  has a nonvanishing section (or equivalently  $SE$  has a section) then  $e(E) = 0$ . (Hint: this is easy.) Deduce that the vector bundle from Problem 2 does not have a nonvanishing section.
- (4) What does the existence of the Gysin sequence imply about when there are fiber bundles  $S^m \rightarrow S^n$  with fiber a sphere  $S^k$ ? (i.e., what relation does it imply among  $m, n, k$ ?)

Suggested review / qualifying exam practice (not to turn in):

- (1) Prove the Thom isomorphism theorem.
- (2) Let  $STS^2$  be the unit sphere bundle of the tangent bundle of  $S^2$ . Use the Mayer-Vietoris sequence to compute the cohomology of  $STS^2$ , and deduce that  $TS^2$  has no nonvanishing section.

More problems to think about but not turn in:

- (1) Call a sphere bundle  $p: E \rightarrow B$  (with fiber  $S^n$ ) *orientable* if for every loop  $\gamma: S^1 \rightarrow B$ ,  $\gamma^*E \cong S^1 \times S^n$  (as bundles over  $S^1$ ). Prove that if  $E \rightarrow B$  is an orientable sphere bundle then the Thom isomorphism holds for the bundle  $DE \rightarrow B$  where  $DE$  is the mapping cone of the projection  $p: E \rightarrow B$  (so  $DE$  is a bundle with fibers  $D^{n+1}$ ). Deduce that the Gysin exists for any orientable sphere bundle.
- (2) Give an example of a non-orientable sphere bundle for which the Gysin sequence does not exist.

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